MANAGERIAL ECONOMICS

Corrado Pasquali

UNIVERSITÀ DEGLI STUDI DI TERAMO

5. QUICK RECAP ON CHOICE UNDER UNCERTAINTY

The sample space

When dice are rolled, we say that the set

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

of all possible outcomes is a sample space.

Events

The events that can result from rolling the dice are identified with the subsets of Ω. Thus the event that the dice shows an even number is the set $E = \{2, 4, 6\}$

Probability measures

A *probability measure* is a function defined on the set S of all possible events.
The number $\mathrm{prob}(E)$ is said to be the probability of the event E.
To qualify as a probability measure, the function $\mathrm{prob} : S \to [0, 1]$ must satisfy three properties.

First property

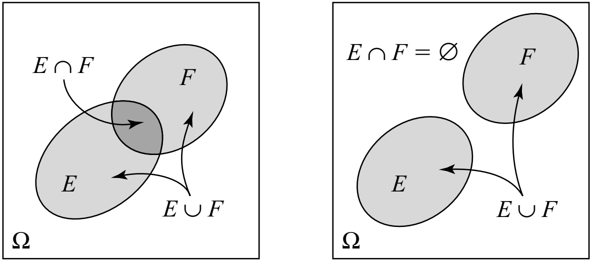
The first property is that $\mathrm{prob}(\emptyset) = 0$. Since $\emptyset$ is the set with no elements, this means that the probability of the impossible event that nothing at all will happen is zero.

Second property

The second property is that $\mathrm{prob}(\Omega) = 1$, which means that the probability of the certain event that something will happen is 1.

Third property

The third property says that the probability that one or the other of two events will occur is equal to the sum of their separate probabilities – provided that the two events can't both occur simultaneously.
The set $E \cap F$ represents the event that both events E and F occur at the same time. So $E \cap F = 0$ means that E and F can't occur simultaneously, as in the Figure:



Third property

The set $E \cup F$ represents the event that at least one of E or F occurs. So the third property can be expressed formally by writing

$$E \cap F = 0 \rightarrow \mathrm{prob}(E \cup F) = \mathrm{prob}(E) + \mathrm{prob}(F)$$

Third property

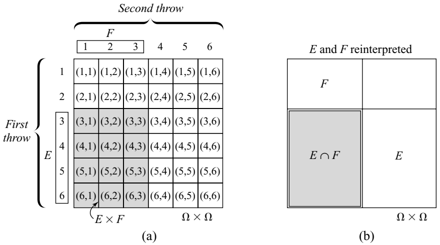
A fair dice is equally likely to show any of its faces when rolled, and so $\mathrm{prob}(1) = \mathrm{prob}(2) = \ldots = \mathrm{prob}(6) = 1/6$. The probability of the event $E = \{2, 4, 6\}$ that an even number will appear is therefore:

$$\mathrm{prob}(E) = \mathrm{prob}(2) + \mathrm{prob}(4) + \mathrm{prob}(6) = 1/6 + 1/6 + 1/6 = 1/2$$

Independent rolls

If A and B are sets, then $A \times B$ is the set of all pairs (a, b) with $a \in A$ and $b \in B$. Next slide shows the sample space $\Omega^2 = \Omega \times \Omega$ obtained when two independent rolls of the dice are observed.

Independent rolls



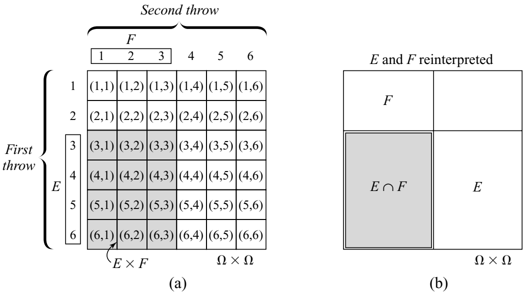
Independent rolls

There are $36 = 6 \times 6$ possible outcomes in $\Omega \times \Omega$. If the two dice are rolled independently, each outcome is equally likely. The probability of each is therefore $1/36$ So the probability of $E \times F$ must be:

$$\mathrm{prob}(E \times F) = 12/36 = 1/3$$

Notice that $\mathrm{prob}(E) = 2/3$ and $\mathrm{prob}(F) = 1/2$. Thus,

$$\mathrm{prob}(E \times F) = \mathrm{prob}(E) \times \mathrm{prob}(F)$$



Independent rolls

The equation

$$\mathrm{prob}(E \times F) = \mathrm{prob}(E) \times \mathrm{prob}(F)$$

holds whenever E and F are independent events. The conclusion is usually expressed as:

$$\mathrm{prob}(E \cap F) = \mathrm{prob}(E)\mathrm{prob}(F)$$

which says that the probability that two independent events will both occur is the product of their separate probabilities.

Conditional probability

After you observe that an event F has happened, your knowledge base changes. The only states of the world that are now possible lie in the set F.

You must therefore replace Ω by F, which is the new world in which your future decision problems will be set.

The new probability $\mathrm{prob}(E|F)$ you assign to an event E after learning that F has occurred is called the conditional probability of E given F.

Conditional probability

For example, we know that $\text{prob}(4) = \frac{1}{6}$ when a fair dice is rolled. If we learn that the outcome was even, this probability must be adjusted.

The event $F = \{2, 4, 6\}$ that the outcome is even contains three equally likely states. The probability of rolling a 4, given that F has occurred, is therefore $\frac{1}{3}$. Thus,

$$\text{prob}(4|F) = \frac{1}{3}$$

The principle on which this calculation is based is embodied in the formula:

$$\text{prob}(E|F) = \text{prob}(E \cap F)/\text{prob}F$$

Lotteries

A bookie may offer you odds of 3:4 against an even number being rolled with a fair dice.

If you take the bet, you win $3 if an even number appears and lose $4 if an odd number appears.

Accepting this bet is equivalent to choosing or accepting to participate in a lottery.

Lotteries

Accepting this bet is equivalent to choosing the lottery L.

The top row shows the possible final outcomes or prizes, and the bottom row shows the respective probabilities with which each prize is awarded.

The lottery M of Figure 3 has three prizes. You have five chances in every twelve of winning the big prize of $24.

L =

\$3	−\$4
$\frac{1}{2}$	$\frac{1}{2}$

(a)

M =

−\$4	\$24	\$3
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{3}$

(b)

Random variables

Lotteries are nothing but *random variables*. A random variable is a function:

$$X : \Omega \mapsto \mathbb{R}$$

Random variables

The lottery L is equivalent to the random variable $X : \Omega \mapsto \mathbb{R}$ defined by:

$$X(\Omega) = \begin{cases} 3 & \text{if } \omega = 0\ 2,\ 4,\ 6 \\ -4 & \text{if } \omega = 1{,}3,\ 5 \end{cases} \tag{1}$$

If you take the bet represented by the random variable X, your probability of winning \$3 is $\mathrm{prob}(X = 3) = \mathrm{prob}(2, 4, 6) = \frac{1}{2}$ Your probability of losing \$4 is $\mathrm{prob}(X = -4) = \mathrm{prob}(1, 3, 5) = \frac{1}{2}$

Expected value

The expectation or expected value $\mathbb{E}X$ of a random variable X is defined by:

$$\mathbb{E}X = \sum k \,\mathrm{prob}(X = k) \tag{2}$$

where the summation extends over all values of k for which $\mathrm{prob}(X = k)$ isn't zero.

Expected value

Your expected dollar winnings in the lottery L are

$$\mathbb{E}(a) = 3 \times \frac{1}{2} + (-4) \times \frac{1}{2} = -\frac{1}{2} \quad (3)$$

L =

\$3	−\$4
$\frac{1}{2}$	$\frac{1}{2}$

(a)

M =

−\$4	\$24	\$3
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{3}$

(b)

If you bet over and over again on the roll of a fair die, winning \$3 when the outcome is even and losing \$4 when the outcome is odd, you are therefore likely to lose an average of about 50 cents per bet in the long run.

Expected value

The expected dollar value of lottery M is:

$$\mathbb{E}(b) = (-4) \times \frac{1}{4} + 24 \times \frac{5}{12} + 3 \times \frac{1}{3} = 10 \quad (4)$$

If you repeatedly paid \$3 for a ticket in this lottery, you would be likely to win an average of about \$7 per trial in the long run.

L =

\$3	−\$4
$\frac{1}{2}$	$\frac{1}{2}$

(a)

M =

−\$4	\$24	\$3
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{3}$

(b)

Risky choices

How do we describe a player's preferences over lotteries that involve more than two prizes?

A naive approach would be to replace all the prizes in the lotteries by their worth to the player in money.

Wouldn't a rational person then simply prefer whichever of two lotteries has the larger dollar expectation?

. . . meet the St.Petersburg paradox!

St. Petersburg paradox

A fair coin is tossed until it shows heads for the first time. If the first head appears on the k^{th} trial, you win $\$2^k$. How much should you be willing to pay in order to participate in this lottery?

prize	\$2	\$4	\$8	\$16	...	$\$2^k$	...
coin sequence	H	TH	TTH	$TTTH$	...	$TT\ldots TH$	...
probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...	$\left(\frac{1}{2}\right)^k$	...

St. Petersburg paradox

Since each toss of the coin is independent, the probability of winning $\$2^k$ is calculated as shown below for the case $k = 4$:

$$\mathrm{prob}(TTTH) = \mathrm{prob}(T) \times \mathrm{prob}(T) \times \mathrm{prob}(T) \times \mathrm{prob}(H) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

St. Petersburg paradox

The expectation in dollars of the St. Petersburg lottery L is therefore:

$$\mathbb{L} = 2\text{prob}(H) + 4\text{prob}(TH) + 8\text{prob}(TTH) + \ldots$$

$$2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \ldots$$

$$1 + 1 + 1 \ldots$$

which implies that the expected dollar value is infinite.

So: should we go sell off all that we own to participate in the lottery?

Expected utility theory

An adequate theory needs to recognize that the extent to which one is willing to bear risk is as much a part of her preference profile.

This is exactly the core of Von Neumann and Morgenstern expected utility theory.

Olga's utility

Suppose that Olga's utility for money is given by the Von Neumann and Morgenstern utility function $u : \mathbb{R}_+ \mapsto \mathbb{R}$ defined by:

$$u(x) = 4\sqrt{x}$$

then, her expected utility for the St. Petersburg lottery L of Figure 6 is given by:

$$\begin{aligned}
\mathbb{E}u(L) &= \tfrac{1}{2}u(2) + \left(\tfrac{1}{2}\right)^2 u(2^2) + \left(\tfrac{1}{2}\right)^3 u(2^3) + \ldots \\
&= 4\left\{\tfrac{1}{2}\sqrt{2} + \left(\tfrac{1}{2}\right)^2\sqrt{2^2} + \left(\tfrac{1}{2}\right)^3\sqrt{2^3} + \ldots\right\} \\
&= \tfrac{4}{\sqrt{2}}\left\{1 + \tfrac{1}{\sqrt{2}} + \left(\tfrac{1}{\sqrt{2}}\right)^2 + \ldots\right\} \\
&= \tfrac{4}{\sqrt{2}-1} \approx 4 \times 3.42
\end{aligned}$$

Olga is thus indifferent between the lottery L and \$$X$ iff their utilities are the same.

So, X is the dollar equivalent of the lottery L iff

$$u(X) = \mathbb{E}u(L)$$

$$4\sqrt{X} \approx 4 \times 3.42$$

$$X \approx (3.242)^2 = 11.70$$

Remember: the "dollar equivalent" is the smallest amount in dollars for which the agent would be willing to forego enjoying the prize.

Risk attitude

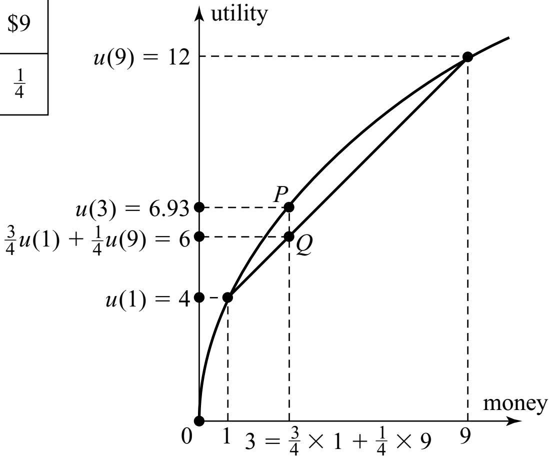
Thus Olga won't pay more than $ 11.70 to participate in the St. Petersburg lottery - which is a lot less than the infinite amount she would pay if her Von Neumann and Morgenstern utility function were $u(x) = x$.

We will see that the reason we get such a different result is that Olga's new Von Neumann and Morgenstern utility function makes her risk averse instead of risk neutral.

Risk attitude

Consider now lottery M:

$$\mathbf{M} = \begin{array}{|c|c|} \hline \$1 & \$9 \\ \hline \frac{3}{4} & \frac{1}{4} \\ \hline \end{array}$$



Risk attitude

The dollar expectation of M is:

$$\mathbb{E}M = \frac{3}{4} \times 1 + \frac{1}{4} \times 9 = 3$$

If Olga's Von Neumann and Morgenstern utility for \$x continues to be $u(x) = 4\sqrt{x}$, then her expected utility for M is:

$$\mathbb{E}u(M) = \frac{3}{4}u(1) + \frac{1}{4}u(9) = \frac{3}{4} \times 4\sqrt{1} + \frac{1}{4} \times 4\sqrt{9} = 6$$

It follows that:

$$u(\mathbb{E}M) = u(3) = 4\sqrt{3} \approx 6.93$$

and so Olga would rather not participate in the lottery if she can have its expected dollar value for certain instead.

If Olga would always sell a ticket for a lottery with money prizes for an amount equal to its expected dollar value, she is risk averse over money.

If she would always buy a ticket for a lottery for an amount equal to its expected dollar value, then she is risk loving. If she is always indifferent between buying and selling, she is risk neutral.