- **ELEMENTS OF SET THEORY**

Set:

A set is a collection, a grouping, a family of objects, called **elements**.

Notation:

- Sets are denoted by capital letters $(A, B, C, \ldots)$
- Elements belonging to sets are denoted by lowercase letters $(a, b, c, \ldots)$

- $a \in A$ means: a belongs to A
- $a \notin A$ means: a does not belong to A

- Cardinality of a set: the number of its elements

Representations of sets

To describe and specify the elements of a set, different representations may be used:

- Roster (tabular) representation
- Graphical representation
- Set-builder (characteristic) representation

Set-builder (characteristic) representation

The set-builder representation consists in defining the elements of a set through a common property, called the characteristic property.

A set is described as a collection of elements satisfying a given property

Some definitions

- **Empty set:** a set with no elements (symbol ∅)
- Two sets are **equal** if they contain the same elements ($A = B$) → each element of one set also belongs to the other set
- Otherwise, the sets are called **different** ($A \neq B$)
- If no element of A belongs to B, the sets are **disjoint**

$$A = \{r, t\}, \qquad B = \{t, r\} \quad \rightarrow \quad A = B$$

$$A = \{a, b, c\}, \qquad B = \{a, d, e\} \quad \rightarrow \quad A \neq B$$

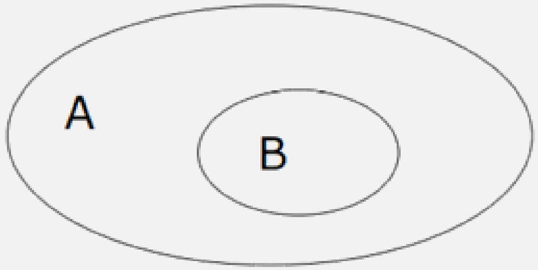
$$A = \{a, b, c\}, \qquad B = \{m, n, t\} \quad \rightarrow \quad A \text{ and } B \text{ disjoint}$$

Set inclusion: subsets

Set B is a **subset** of A if every element of B is also an element of A:

$$A \supseteq B \quad or \quad B \subseteq A$$

$$\forall x \in B, x \in A$$



if there exists at least one element of A not belonging to B, then B is defined as a **proper subset** of A :

$$A \supset B \quad or \quad B \subset A$$

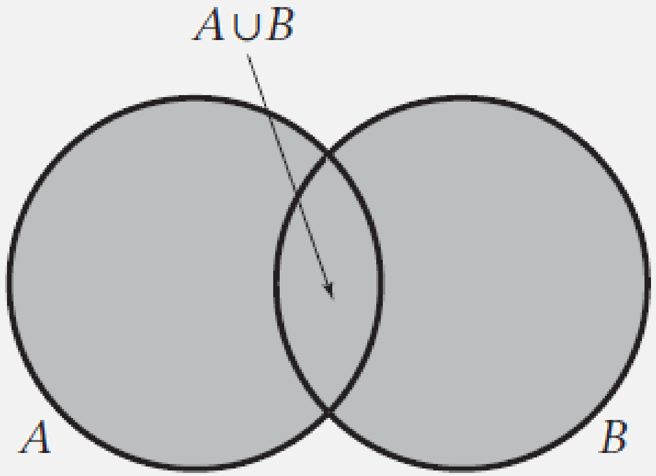
$$\{x \in \mathbb{N} : x \ even\} \subseteq \mathbb{N} \ , \{0,2\} \subset \{0,2,4,6\}$$

$$B \subsetneq A \quad means \quad B \subset A \text{ and } B \neq A$$

Operations on sets – Union

Given the sets A and B , the **union set** is the set of elements belonging to at least one of the two sets:

$$A \cup B = \{x : x \in A \textbf{ or } x \in B\}$$

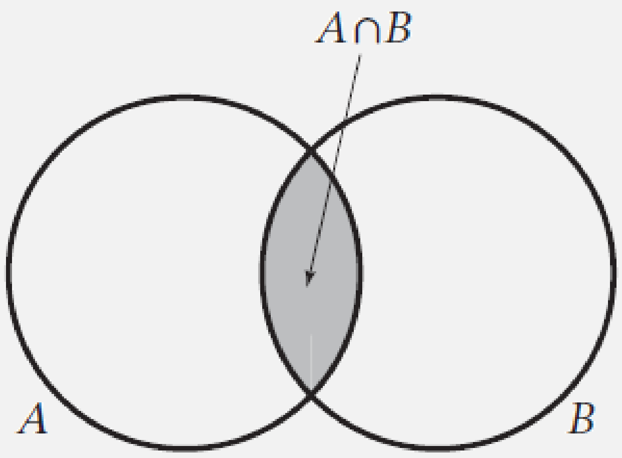


Operations on sets – Intersection

The **intersection set** is the set of elements common to both sets:

$$A \cap B = \{x : x \in A \textbf{ and } x \in B\}$$

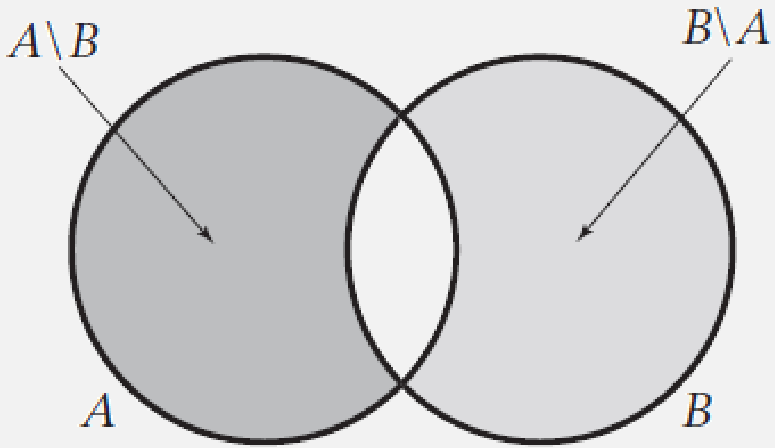
➢ if $A \cap B = \emptyset$, then A and B are **disjoint**



Operations on sets – Difference

The **difference set** consists of the elements belonging to A but not to B

$$A - B = A\backslash B = \{x : x \in A \textbf{ and } x \notin B\}$$



If $B \subset A$, then:

$$B^C = A\backslash B = \{x : x \in A \textbf{ and } x \notin B\}$$

where B^C is the **complement set** of B (in A)

Numerical sets

$\mathbb{N} = \{0,1,2,3, \dots\}$ the set of **natural numbers**

$\mathbb{N}^* = \{1,2,3, \dots\}$ the set of **natural numbers different from zero**

$\mathbb{Z} = \{\dots, -2, -1,0,1,2, \dots\}$ the set of **integers**

$\mathbb{Q} = \{\frac{n}{m} : n, m \in \mathbb{Z}, m \neq 0\}$ the set of **rational numbers** (a rational number, written in its decimal form, after the comma shows a finite decimal expansion, or an infinite periodic one)

$\mathbb{R}$ the set of **real numbers**→ all the numbers that, written in their decimal form, show an infinite decimal expansion, also non-periodic

The set $\mathbb{N}$

$\mathbb{N} = \{0,1,2,3, \dots\}$

In $\mathbb{N}$ it is possible to establish an order relation / ordering:

- if there exists a natural number x that added to $a \in \mathbb{N}$, gives the natural number b, then a is less than b $a \leq b$ (if $a \leq b$ and $a \neq b$, then $a < b$)

- Similarly, if a is greater than b $a \geq b$ (if $a \geq b$ and $a \neq b$, then $a > b$)

This total order relation satisfies the following properties:

- $\forall n \in \mathbb{N}, n \leq n$ (**reflexive property / reflexivity**)
- $\forall n, m \in \mathbb{N}$, if $n \leq m$ *and* $m \leq n$, then $n = m$ (**antisymmetric property / antisymmetry**)
- $\forall n, m, p \in \mathbb{N}$, if $n \leq m$ *and* $m \leq p$, then $n \leq p$ (**transitive property / transitivity**)

The set $\mathbb{N}$

$\mathbb{N} = \{0,1,2,3,\dots\}$

- By adding or multiplying natural numbers, one always obtains natural numbers.
- The number 0 is the neutral element for addition in $\mathbb{N}$:

$$\forall n \in \mathbb{N}, n + 0 = n, 0 + n = n$$

- The number 1 is the neutral element for multiplication in $\mathbb{N}$:

$$\forall n \in \mathbb{N}, n \cdot 1 = n, 1 \cdot n = n$$

The set $\mathbb{Z}$

$\mathbb{Z} = \{\dots, -3, -2, -1,0,1,2,3, \dots\}$

Formally, given $a, b \in \mathbb{N}$, the equation $a + x = b$ may have no solution in $\mathbb{N}$
It is therefore necessary to enlarge the set $\mathbb{N}$, defining the set $\mathbb{Z}$

- In $\mathbb{Z}$ the same order relation holds
- In $\mathbb{Z}$ the number 0 remains the neutral element for addition
- The solution of the equation $a + x = 0$ with $a \in \mathbb{N}$, is denoted by $-a$ → the element which, when added to a yields 0 is called the **additive inverse** (or **opposite**) of a

The set $\mathbb{Q}$

$\mathbb{Q} = \{\frac{n}{m} : n, m \in \mathbb{Z}, m \neq 0\}$

Let us try to invert the operation of multiplication in $\mathbb{Z}$.
Consider for $a, b \in \mathbb{Z}$ the equation $ax = b$, with $a \neq 0$
This equation has a solution in $\mathbb{Z}$ if and only if b is a multiple of a

It is therefore necessary to enlarge the set $\mathbb{Z}$, by defining the set $\mathbb{Q}$

$\mathbb{Q}$ is the set of ordered pairs (n, m) with $n, m \in \mathbb{Z}$ and $m \neq 0$.

- For every $x \in \mathbb{Q}, x \neq 0$ there exists an element, denoted by $1/x \in \mathbb{Q}$ such that $x \cdot \left(\frac{1}{x}\right) = 1$.
  This element is called the **reciprocal** or **multiplicative inverse** of x

The set $\mathbb{R}$

The most important properties of the set $\mathbb{R}$:

1. Arithmetic operations, together with their properties defined on rational numbers, extend to real numbers
2. Defining $\mathbb{R}_+$ e $\mathbb{R}_-$ as the subsets of positive and negative real numbers, respectively, we have:
$$\mathbb{R} = \mathbb{R}_+ \cup \{0\} \cup \mathbb{R}_-$$
Therefore, formally we can write:
$$x \leq y \iff y - x \in \mathbb{R}_+ \cup \{0\}, \qquad x < y \iff y - x \in \mathbb{R}_+$$

   Further properties of the order relation:
   - $\forall x, y, z \in \mathbb{R}, x < y \Rightarrow x + z < y + z$
   - $\forall x, y, z \in \mathbb{R}, x < y, z > 0 \Rightarrow x \cdot z < y \cdot z$
   - $\forall x, y, z \in \mathbb{R}, x < y, z < 0 \Rightarrow x \cdot z > y \cdot z$
   - $\forall x, y \in \mathbb{R}, 0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$
3. The set of real numbers is complete: to every point on the **real line** there corresponds one and only one real number.

Numerical sets

Among numerical sets, the following proper inclusions hold:

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

These inclusions are strict, because:

- there exist integers that are not natural numbers (negative integers);
- there exist rational numbers that are not integers (proper fractions);
- there exist real numbers that are not rational (irrational numbers).

In $\mathbb{N}$ there is neither an additive inverse nor a multiplicative inverse for any number → in $\mathbb{N}$ one can perform addition and multiplication, but in general it is not possible to perform the inverse operations of subtraction and division.

With the exception of 1, in $\mathbb{Z}$ no number has a multiplicative inverse → in $\mathbb{Z}$ subtraction is generally possible, whereas division is not.

Maximum, minimum

Let A be a non-empty set of real numbers.

The maximum of A, if it exists, is defined as the number M that belongs to A and is greater than or equal to every other element of the set A

$$A \subset \mathbb{R} \qquad M \text{ maximum of } A \Leftrightarrow \begin{cases} M \in A \\ \forall a \in A, \;\; M \geq a \end{cases}$$

Note. Not all sets of real numbers admit a maximum or a minimum.

Maximum, minimum

Let A be a non-empty set of real numbers.

The minimum of A, if it exists, is defined as the number m that belongs to A and is smaller than or equal to every other element of the set A

$$A \subset \mathbb{R} \qquad m \text{ minimum of } A \Leftrightarrow \begin{cases} m \in A \\ \forall a \in A, \; m \leq a \end{cases}$$

Example: the set $A = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ has , $\max(A) = \frac{1}{2}$, but does not have a minimum

Maximum, minimum

Example 1.

Let $A \subset \mathbb{R}$ be the set of positive real numbers, defined by

$$A = \left\{ x \in \mathbb{R} : x > 0 \right\}$$

Then the set A admits neither a minimum nor a maximum, that is,

$$\nexists \min A \; and \; \nexists \max A$$

Indeed, althgough zero is a lower bound of A it does not belong to the set, since $0 \notin A$

Example 2. Consider the set $\mathbb{N}$ of natural numbers. One has

$$\exists \min \mathbb{N} = 1$$

whereas

$$\nexists \max \mathbb{N}$$

Upper and lower bounds

Let A be a non-empty set of real numbers.

A real number M is called upper bound of A, if there exists at least on real number greater or equal to all the elements of A

$$a \leq M, \forall a \in A$$

Let A be a non-empty set of real numbers.

A real number m is called lower bound of A, if there exists at least on real number less or equal to all the elements of A

$$a \geq m, \forall a \in A$$

The set A is said to be **bounded above** if it admits at least one upper bound, that is, if:

$$\exists\, r \in \mathbb{R} : r \geq x, \forall x \in A$$

The set A is said to be **bounded below** if it admits at least one upper bound, that is, if:

$$\exists\, r \in \mathbb{R} : r \leq x, \forall x \in A$$

The set A is said to be **bounded** if it admits at least both an upper and a lower bound:

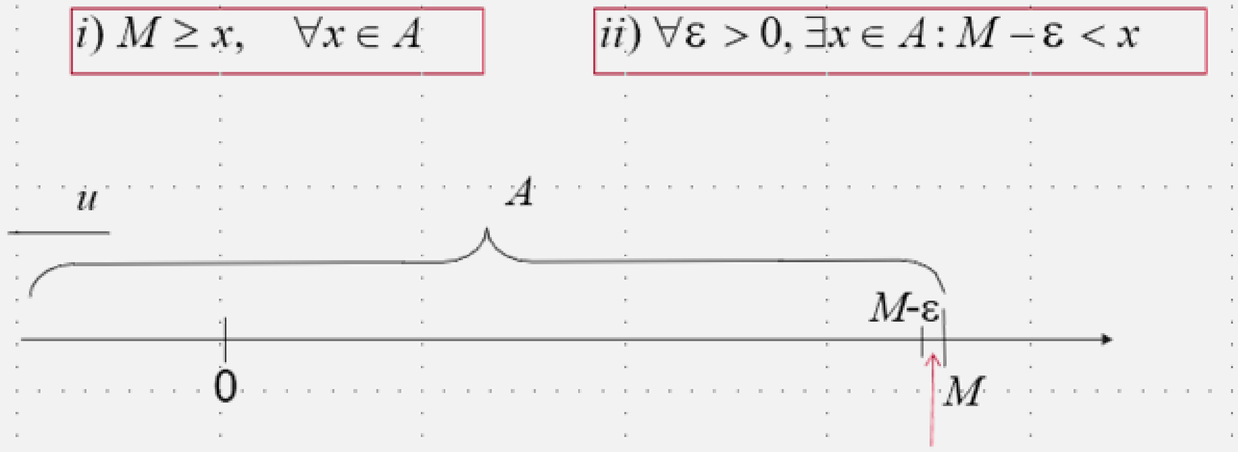
$$\exists l, L \in R : l \leq a \leq L, \forall a \in A$$

Supremum and infimum

Let A be a non-empty subset of the real numbers that is bounded above (that is, it admits upper bounds). Then, a real number $M \in \mathbb{R}$ is called the **supremum** (or **least upper bound**) of A if it is the smallest of all upper bounds of A:

$$M = \sup A \Leftrightarrow \begin{cases} M \geq x, \forall x \in A \\ \forall \varepsilon > 0, \exists x \in A : M - \varepsilon < x \end{cases}$$

The supremum follows these properties:

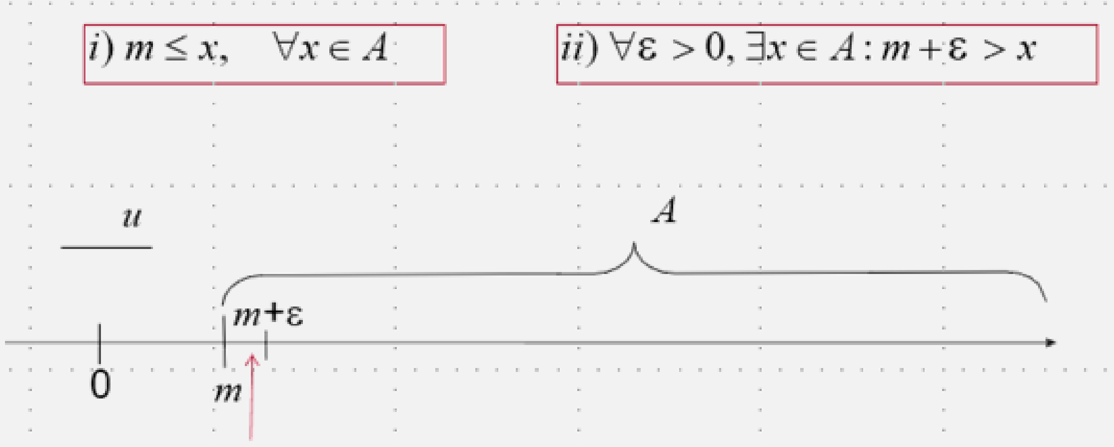


Supremum and infimum

Let A be a non-empty subset of the real numbers that is bounded above (that is, it admits upper bounds). Then, a real number $m \in \mathbb{R}$ is called the **infimum** (or **greatest lower bound**) of A if it is the greatest of all lower bounds of A:

$$m = \inf A \Leftrightarrow \begin{cases} m \leq x, \forall x \in A \\ \forall \varepsilon > 0, \exists x \in A : m + \varepsilon > x \end{cases}$$

The infimum follows these properties:



Intervals

An empty I of the real number line is called an **interval** if $\forall x, y \in I$ every point lying between x and y also belongs to I

$A = \{x \in \mathbb{R} : x \geq 6\}$ → is an interval

$B = \{x \in \mathbb{R} : x \neq 0\}$ → is not an interval

Among the **bounded intervals**, for $a, b \in \mathbb{R} : a < b$, we have:

- **Closed interval** $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- **Right-open interval** $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- **Left-open interval** $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- **Open interval** $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

All these intervals have lower endpoint a and upper endpoint b.

Intervals

Among **unbounded intervals**, we have:

- Closed interval unbounded above $[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}$
- Open interval unbounded above $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$
- Closed interval unbounded below $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$
- Open interval unbounded below $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$

Neighborhood

Given a point $x \in \mathbb{R}$, a neighborhood of x is any subset of $\mathbb{R}$ that contains an open interval $(x - \varepsilon, x + \varepsilon)$, with $\varepsilon > 0$