- **EQUATIONS AND INEQUALITIES OF FIRST AND SECOND DEGREE**

Equation:

Solving an equation means finding the unknown value, or values, for which the two algebraic expressions yield the same result.

A set of values that, when substituted for the unknowns, makes an equation true is called the **solution set** or the **set of roots**.

Therefore, solving an equation means explicitly determining the set of all its solutions.

Example.

Solving the equation

$$5x + 3 = 2x + 4$$

means finding the value of x for which

The left-hand side $5x + 3$ and the right-hand side $2x + 4$

Have the same value

First-Degree Equations

EQUIVALENCE PRINCIPLES

The solution set of an equation does not change if:

- The same quantity (number or unknown) is added to both sides of the equation.
  Consequently, a term can be moved from one side to the other provided its sign is changed, and identical terms appearing on both sides can be eliminated.
- Both sides of the equation are multiplied by the same nonzero number.
  Consequently, the sign of both sides of the equation may be changed (both sides, not just one).

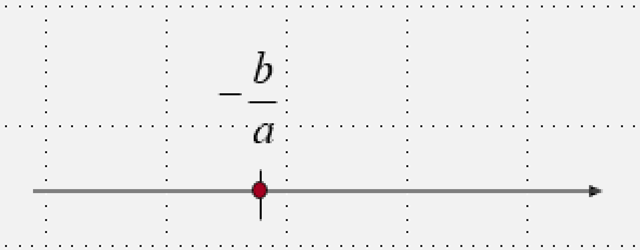
Linear Algebraic Equations

A **first-degree (or linear)** algebraic equation is an equation in which the highest degree of the unknown is one and which, after suitable simplifications, can be written in the **standard form**:

$$ax + b = 0$$

where $a \neq 0$ and $a, b \in \mathbb{R}$.

Solving for x gives:



$$x = -\frac{b}{a}$$

UNIQUE SOLUTION!

Special cases

Given the equation:

$$ax + b = 0 \text{ with } a, b \in \mathbb{R}$$

If $a = 0$, one can distinguish two cases:

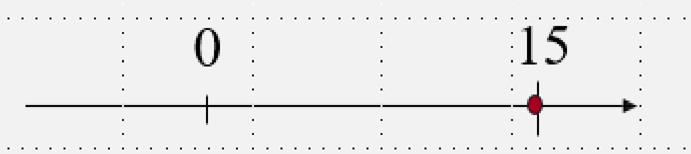
- $0 \cdot x + b = 0 \quad con\ b \neq 0$ → **inconsistent equation (no solution)**
- $0 \cdot x + b = 0 \quad con\ b = 0$ → **indeterminate equation (infinitely many solutions)**

Exercise 1. solve the following equation: $x + 6 = 21$

Adding to both sides number -6, the root set does not change:

$$x + \cancel{6} - \cancel{6} = 21 - 6$$

The solution is $x = 15$



Exercise 2. solve the following equation : $4x - 3 = -9$

Adding to both sides number 3, the root set does not change :

$$4x - 3 + 3 = -9 + 3$$

Dividing both sides by number 4, the root set does not change :

$$\frac{4x}{4} = -\frac{6}{4}$$

The solution is $x = -\frac{6}{4}$

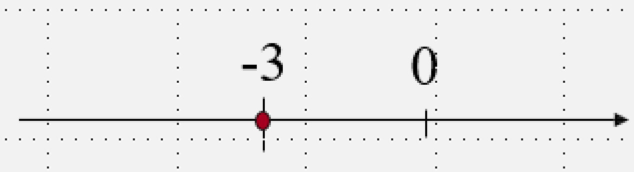
Exercise 3. solve the following equation : $(x+1)^2 - x + 2 = x^2$

Solving the square of a binomial in the left-hand side of the equation: $x^2 + 2x + 1 - x + 2 = x^2$

Adding $-x^2$ e -3 to both sides, the equation becomes:

$$2x + 1 - x + 2 - 3 = -3$$

The solution is $x = -3$



Second-Degree Algebraic Equations

A second-degree (or quadratic) equation is an algebraic equation in one unknown x, with maximum degree equal to 2, and can be written in the **standard form**:

$$ax^2 + bx + c = 0,$$
$$a \neq 0, a, b, c \in \mathbb{R}$$

Example

The equation $3x^2 + 2x - 5 = 0$ is a second-degree algebraic equation written in its standard form with $a = 3, b = 2, c = -5$

Solutions of Quadratic Equations

In the real number field, a quadratic equation may have:

- two distinct real solutions,
- two coincident real solutions,
- no real solutions.

Example

$$x = 1 \text{ and } x = 2$$

Are solutions of the equation:

$$x^2 - 3x + 2 = 0$$

Discriminant

The **discriminant** of a quadratic equation

$$ax^2 + bx + c = 0$$

Is defined as:

$$\Delta = b^2 - 4ac$$

Example

For the equation

$$3x^2 + 2x - 5 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 2^2 - 4(3)(-5) = 4 + 60 = 64$$

Nature of the solutions

In contrast to first-degree equations, which always admit a unique solution, a second-degree equation may admit either no real solutions or two real solutions, which may coincide.

The existence and the qualitative nature of the real solutions are determined by the sign of the discriminant

$$\Delta = b^2 - 4ac$$

- if $\Delta = b^2 - 4ac > 0$

the equation admits **two distinct real solutions**, denoted by x_1 and x_2

→ And the following factorization holds:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

- if $\Delta = b^2 - 4ac = 0$

the equation admits **two coincident real solutions**, denoted by x_0

→ And the following factorization holds:

$$ax^2 + bx + c = a(x - x_0)^2$$

- if $\Delta = b^2 - 4ac < 0$

the equation admits no real solutions

Nature of the solutions

- if $\Delta = b^2 - 4ac > 0$

the equation admits **two distinct real solutions**

- if $\Delta = b^2 - 4ac = 0$

the equation admits **two coincident real solutions**

if $\Delta = b^2 - 4ac \geq 0$ the two solutions are given by the formula:

$$x_{1,2} = -\frac{b \pm \sqrt{\Delta}}{2a}$$

Exercise. Solve the following equation: $x^2 + x - 2 = 0$

We compute the value of the discriminant:

$$\Delta = b^2 - 4ac = 1 + 8 = 9 > 0$$

Therefore, the equation admits two distinct real solutions.
We now determine them:

$$x_{1,2} = -\frac{1 \pm \sqrt{9}}{2 \cdot 1} =$$

$$x_1 = -\frac{4}{2} = -2$$

$$x_2 = \frac{2}{2} = 1$$

Exercise. Solve the following equation: $2x^2 - 5x + 3 = 0$

We compute the value of the discriminant:

$$\Delta = b^2 - 4ac = 25 - 4 \cdot 3 \cdot 2 = 1 > 0$$

Therefore, the equation admits two distinct real solutions.
We now determine them:

$$x_{1,2} = -\frac{5 \pm \sqrt{1}}{2 \cdot 2} =$$

$$x_1 = \frac{3}{2}$$

$$x_2 = 1$$

Exercise. Solve the following equation: $-2x^2 + x + 1 = 0$

Before computing the discriminant, we change the sign to both the sides of the equation:

$$2x^2 - x - 1 = 0$$

We compute the value of the discriminant :

$$\Delta = b^2 - 4ac = 1 + 8 = 9 > 0$$

Therefore, the equation admits two distinct real solutions.
We now determine them:

$$x_{1,2} = -\frac{1 \pm \sqrt{9}}{2 \cdot 2} = \begin{cases} x_1 = -\frac{1}{2} \\ x_2 = 1 \end{cases}$$

Exercise. Solve the following equation: $2(x+1)-(4-2x)=x^2+3$

We reduce the equation to its standard form:

$$2x+2-4+2x=x^2+3 \;\rightarrow -x^2+4x-5=0 \;\rightarrow x^2-4x+5=0$$

We compute the value of the discriminant :

$$\Delta = b^2-4ac=16-20=-4<0$$

Therefore, the equation admits no real solutions.

Exercise. Solve the following equation: $x(x^2 - 3x) + 2 = x^3$

We reduce the equation to its standard form:

$$x^3 - 3x^2 + 2 = x^3 \rightarrow -3x^2 + 2 = 0 \ \rightarrow 3x^2 - 2 = 0$$

This is an incomplete equation that can be solved:

$$3x^2 - 2 = 0 \ \rightarrow x^2 = \frac{2}{3}$$

$$x_1 = \sqrt{\frac{2}{3}}$$

$$x_2 = -\sqrt{\frac{2}{3}}$$

Therefore, the equation admits two distinct real solutions.

Exercise. Solve the following equation: $(x+1)^3 + \frac{(x-1)(2-x)}{2} = (x^2-1)x$

We reduce the equation to its standard form :

$$x^3 + 3x^2 + 3x + 1 + \frac{2x - x^2 - 2 + x}{2} = x^3 - x \ \rightarrow 5x^2 + 11x = 0$$

This is an incomplete equation that can be solved:

$$5x^2 + 11x = 0 \ \rightarrow x(5x + 11) = 0 \ \rightarrow x = 0, x = -\frac{11}{5}$$

Therefore, the equation admits two distinct real solutions.

Irrational Equations

Irrational equations with a square root

- With a polynomial on the right-hand side: $\sqrt{A} = \mathrm{B} \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A = B^2 \end{cases}$
- With a positive number n on the right-hand side: $\sqrt{A} = n \rightarrow A = n^2$
- With a negative number $-n$ on the right-hand side: $\sqrt{A} = -n \rightarrow$ no solution
- With zero on the right-hand side: $\sqrt{A} = 0 \rightarrow A = 0$

Irrational Equations

Irrational equations with two square roots

- $\sqrt{A} = \sqrt{\mathrm{B}} \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A = B \end{cases}$

- $\sqrt{A} + \sqrt{B} = C \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ \left(\sqrt{A} + \sqrt{B}\right)^2 = C^2 \rightarrow 2\sqrt{AB} = C^2 - A - B^{*} \end{cases}$

* *The solution procedure for irrational equations with a single square root is then applied*

Irrational equations

Irrational equations with cube roots

- $\sqrt[3]{A} = B \;\rightarrow A = B^3$

- $\sqrt[3]{A} = \sqrt[3]{B} \;\rightarrow A = B$ *

**To isolate an irrational equation involving cube roots, it is sufficient to isolate the root(s) and raise both sides of the equation to the third power.*

Absolute value equations

Definition

The absolute value of a real number x is defined as:

- x if x is greater or equal to zero
- $-x$ if x is less than zero

$$|x| \rightarrow \begin{cases} x \geq 0 \\ x \end{cases} \vee \begin{cases} x < 0 \\ -x \end{cases}$$

Absolute Value Equations

Equations with a single absolute value

- With a polynomial on the right-hand side: $|A| = B \rightarrow \begin{cases} A \geq 0 \\ A = B \end{cases} \vee \begin{cases} A < 0 \\ A = -B \end{cases}$

- With a positive number n on the right-hand side: $|A| = n \rightarrow A = n \vee A = -n$

- With a negative number $-n$ on the right-hand side: $|A| = -n \rightarrow$ no solution

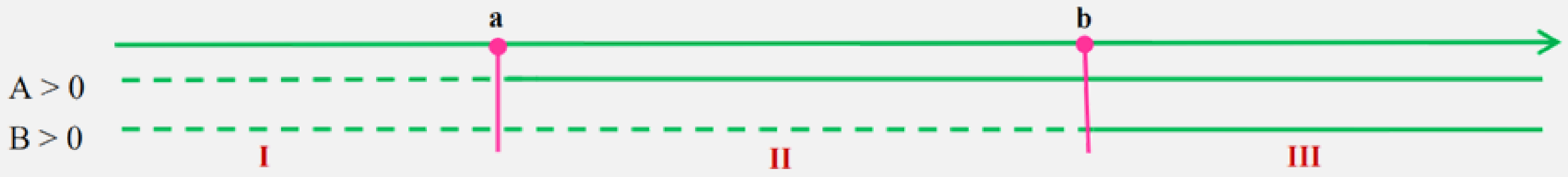
- With zero on the right-hand side: $|A| = 0 \rightarrow A = 0$

Absolute Value Equations

Equations with two or more absolute values

$|A| + |B| = C$ $\rightarrow$ the signs of the expressions A and B are analyzed

- The inequalities $A > 0$ and $B > 0$ are solved and, denoting for instance their solution by $x > a$ and $x > b$, they are represented graphically



- From the analysis of the graph, the equation is decomposed into the following systems.:

$$\text{I} \begin{cases} x < a \\ -A - B = C \end{cases} \quad \vee \quad \text{II} \begin{cases} a \le x \le b \\ A - B = C \end{cases} \quad \vee \quad \text{III} \begin{cases} x > b \\ A + B = C \end{cases}$$

Equations with Two or More Absolute Values: Example

$$|2x - 1| + |3 - x| = 4x + 4$$

$$|2x-1| = \begin{cases} 2x-1 & \text{se } 2x-1 \geq 0 \\ -(2x-1) & \text{se } 2x-1 < 0 \end{cases} \Rightarrow |2x-1| = \begin{cases} 2x-1 & \text{se } x \geq \frac{1}{2} \\ -2x+1 & \text{se } x < \frac{1}{2} \end{cases}$$

$$|3-x| = \begin{cases} 3-x & \text{se } 3-x \geq 0 \\ -(3-x) & \text{se } 3-x < 0 \end{cases} \Rightarrow |3-x| = \begin{cases} 3-x & \text{se } x \leq 3 \\ -3+x & \text{se } x > 3 \end{cases}$$

	$x < \frac{1}{2}$	$\frac{1}{2} \le x \le 3$	$x > 3$
$\lvert 2x-1\rvert =$	$-2x+1$	$2x-1$	$2x-1$
$\lvert 3-x\rvert =$	$3-x$	$3-x$	$-3+x$

Following this approach, the given equation is equivalent to the following three mixed systems:

$$\begin{cases} -2x+1+3-x = 4x+4 \\ x < \frac{1}{2} \end{cases} \cup \begin{cases} 2x-1+3-x = 4x+4 \\ \frac{1}{2} \leq x \leq 3 \end{cases} \cup \begin{cases} 2x-1-3+x = 4x+4 \\ x > 3 \end{cases}$$

$$\begin{cases} x = 0 \\ x < \frac{1}{2} \end{cases} \cup \begin{cases} x = -\frac{2}{3} \\ \frac{1}{2} < x < 3 \end{cases} \cup \begin{cases} x = -8 \\ x > 3 \end{cases}$$

By comparing the solutions of each equation with the conditions imposed by the corresponding systems, it is observed that the second and the third systems are inconsistent, whereas the first system yields the solution

Inequalities

In mathematics, an inequality is a relation of order between two expressions containing unknowns.

Solving an inequality means determining **the set of values** which, when assigned to the unknowns, make the inequality true.

Typically, the solutions of an inequality consist of one or more numerical sets, called **intervals**.

Inequalities

EQUIVALENCE PRINCIPLES

The solution set of an inequality does not change if:

- the same quantity (number or unknown) is added to both sides of the inequality;

consequently, the same term can be eliminated from both sides or moved from one side to the other by changing its sign;

- both sides of the inequality are multiplied or divided by the same quantity, provided that the direction of the inequality is reversed when this quantity is negative;

consequently, all terms on both sides may have their sign changed, provided that the direction of the inequality is also reversed.

Algebraic Inequalities of Degree n

For $n \in \mathbb{N}$, in the unknown x:

$$p(x) > 0 \text{ and } -p(x) < 0$$

are **equivalent inequalities** →

By changing the sign of all coefficients of the polynomial $p(x)$ and reversing the direction of the inequality, one obtains an inequality equivalent to the original one.

Example. Given the fourth-degree inequality:

$$-5x^4 + 3x^3 - x^2 + 1 \leq 0$$

The equivalent inequality is:

$$5x^4 - 3x^3 + x^2 - 1 \geq 0$$

First-Degree Algebraic Inequalities

An inequality is said to be first-degree (or linear) if, after suitable simplifications, it can be written in the form:

$$ax + b \geq 0 \qquad \text{or } > 0, \leq 0, < 0$$

where $a \neq 0$ and $\mathrm{a}, b \in \mathbb{R}$.

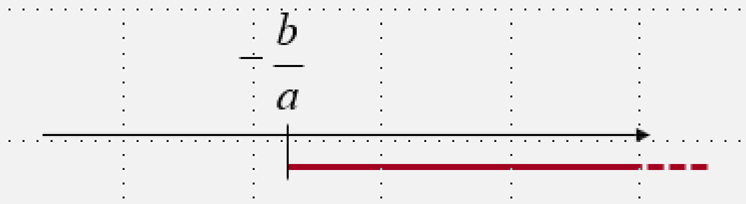
Recalling that the inequalities

$$ax + b > 0 \text{ and } -ax - b < 0 \text{ with } a \neq 0$$

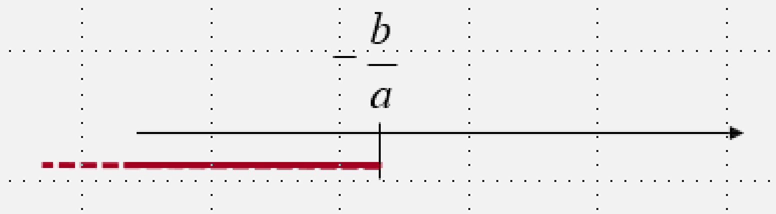
are **equivalent**, the resolution of first-degree inequalities can always be reduced to the case in which the coefficient a is positive.

Case $a > 0$ (with $a \neq 0$ and $a, b \in \mathbb{R}$)

- $ax + b > 0 \rightarrow ax > -b \rightarrow x > -\frac{b}{a}$



- $ax + b < 0 \rightarrow ax < -b \rightarrow x < -\frac{b}{a}$



A first-degree inequality always admits infinitely many solutions given by:

$$x > -\frac{b}{a}$$

$$x < -\frac{b}{a}$$

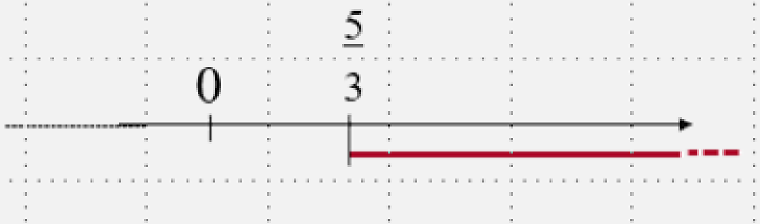
Exercise. Solve the following inequality: $-3x < -5$

The sign of both sides can be changed, reversing the direction of the inequality:

$$3x > 5$$

Multiplying both sides by $1/3 > 0$, one obtains the solution:

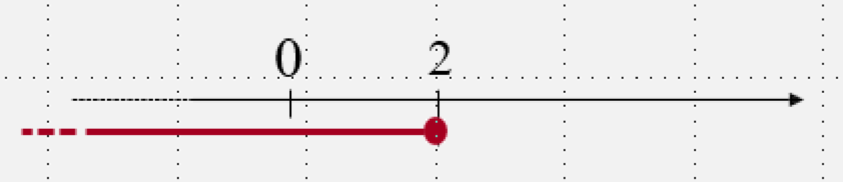
Set of solutions $\quad x > \frac{5}{3}$



Exercise. Solve the following inequality: $5x - 10 \leq 0$

Solution:

$$5x \leq 10 \; \rightarrow x \leq \frac{10}{5} \; \rightarrow x \leq 2$$

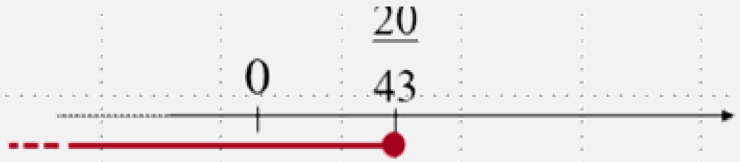


Exercise. Solve the following inequality : $\frac{x-2}{3} + \frac{x}{4} \leq 1 - 3x$

Solution: By computing the least common multiple of the denominators, the inequality is rewritten in an equivalent form:

$$\frac{4x - 8 + 3x}{12} \leq \frac{12 - 36x}{12} \rightarrow 4x - 8 + 3x \leq 12 - 36x \rightarrow 43x \leq 20$$

$$x \leq \frac{20}{43}$$



Set of solutions

Second-Degree Algebraic Inequalities

A second-degree (or quadratic) inequality is an algebraic inequality in one unknown x, whose maximum degree is equal to 2 and which can be reduced to the form:

$$ax^2 + bx + c > 0$$

$$(ax^2 + bx + c < 0)$$

$$a \neq 0, a, b, c \in \mathbb{R}$$

Second-Degree Algebraic Inequalities

By applying the equivalence principles, it should be recalled that, for $a \neq 0, a, b, c \in \mathbb{R}$:

$$ax^2 + bx + c > 0 \text{ and } -ax^2 - bx - c < 0$$

Are equivalent

Therefore, when solving second-degree inequalities, it is always possible to reduce the problem to the case in which the coefficient a is positive.

Second-Degree Algebraic Inequalities – Solutions

$$ax^2 + bx + c > 0 \text{ or } (ax^2 + bx + c < 0)$$

$$a \neq 0, a, b, c \in \mathbb{R}$$

- Step 1:Verify the sign of the coefficient a. If $a < 0$, change the sign of all terms and reverse the direction of the inequality.

- Step 2: Determine the real roots x_1 and x_2 (if they exist) of the associated quadratic equation:

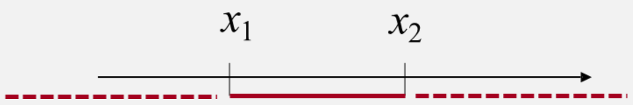
$$x_{1,2} = -\frac{b \pm \sqrt{\Delta}}{2a} \text{ with } \Delta = b^2 - 4ac$$

Second-Degree Algebraic Inequalities – Solutions

CASE I : $\Delta > 0, a > 0$ → Two distinct real solutions

- $ax^2 + bx + c > 0 \Leftrightarrow x < x_1,\ x > x_2$

- $ax^2 + bx + c < 0 \Leftrightarrow x_1 < x < x_2$



CASE I : $\Delta > 0, a > 0$ → Two distinct real solutions

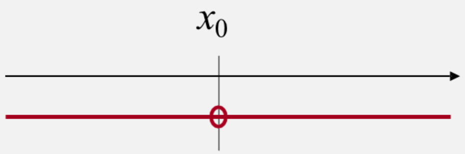
- $ax^2 + bx + c \geq 0 \Leftrightarrow x \leq x_1,\ x \geq x_2$

- $ax^2 + bx + c \leq 0 \Leftrightarrow x_1 \leq x \leq x_2$

Second-Degree Algebraic Inequalities – Solutions

CASE II: $\Delta = 0, a > 0$ → Two coincident real solutions

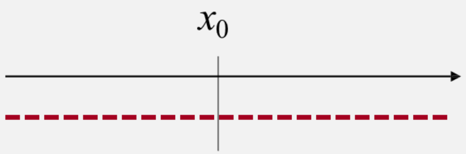
- $ax^2 + bx + c > 0 \Rightarrow \forall x \in R, x \neq x_0$



if the trinomial admits two coincident solutions equal to x_0
one can write:

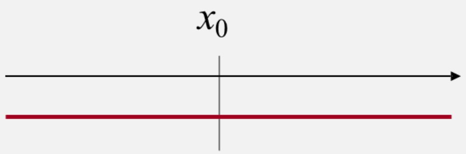
$$ax^2 + bx + c = a(x - x_0)^2$$

- $ax^2 + bx + c < 0 \Rightarrow \nexists x \in R$



CASE II: $\Delta = 0, a > 0$ → Two coincident real solutions

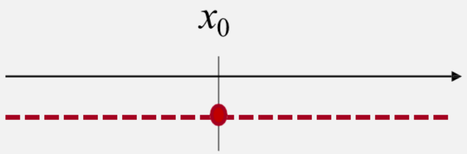
- $ax^2 + bx + c \geq 0 \Rightarrow \forall x \in R$



if the trinomial admits two coincident solutions equal to x_0
one can write:

$$ax^2 + bx + c = a(x - x_0)^2$$

- $ax^2 + bx + c \leq 0 \Rightarrow x = x_0$



Second-Degree Algebraic Inequalities – Solutions

CASE III $: \Delta < 0, a > 0$ → No real solution

- $ax^2 + bx + c > 0 \Rightarrow \forall x \in R$

- $ax^2 + bx + c < 0 \Rightarrow \nexists x \in R$

CASE III $: \Delta < 0, a > 0$ → No real solution

- $ax^2 + bx + c \geq 0 \Rightarrow \forall x \in R$

- $ax^2 + bx + c \leq 0 \Rightarrow \nexists x \in R$

Exercise. Solve the following inequality: $-x^2 + 12x - 11 \geq 0$

Solution: Change the sign of all terms and reverse the direction of the inequality:

$$x^2 - 12x + 11 \leq 0$$

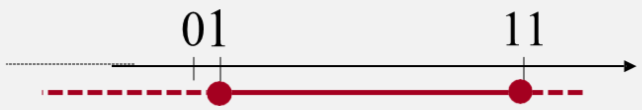
Compute the discriminant $\Delta = b^2 - 4ac = 144 - 44 = 100$

The associated equation admits two distinct real solutions:

$$x_{1,2} = \frac{12 \pm \sqrt{100}}{2} = \frac{12 \pm 10}{2} = \begin{cases} 1 \\ 11 \end{cases}$$

Based on the direction of the inequality and the sign of the discriminant, the solution set is:

$$1 \leq x \leq 11$$



Exercise. Solve the following inequality: $x^2 - 10x + 25 > 0$

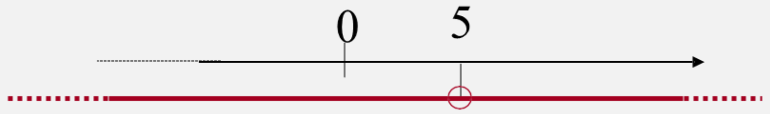
Solution: Compute the discriminant $\Delta = b^2 - 4ac = 100 - 100 = 0$

The associated equation admits two coincident real solutions :

$$x_{1,2} = \frac{10 \pm \sqrt{0}}{2} = \frac{10}{2} = 5$$

Based on the direction of the inequality and the sign of the discriminant, the solution set is :

$$\forall x \in \mathbb{R}, \text{with } x \neq 5$$



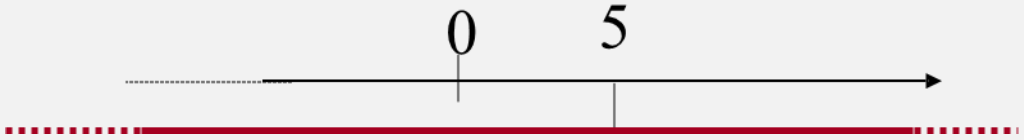
Indeed, the inequality can be written as: $(x-5)^2 > 0$

If the inequality were: $x^2 - 10x + 25 \geq 0$

$$\Delta = b^2 - 4ac = 100 - 100 = 0$$

$$x_{1,2} = \frac{10 \pm \sqrt{0}}{2} = \frac{10}{2} = 5$$

Solution: $\forall x \in \mathbb{R}$



In fact, it is true that: $(x-5)^2 \geq 0$ always!

Exercise. Solve the following inequality: $-12x^2 - 5x \geq 2$

Solution: Change the sign of all terms and reverse the direction of the inequality:

$$12x^2 + 5x + 2 \leq 0$$

Compute the discriminant $\Delta = b^2 - 4ac = 25 - 96 = -71 < 0$

If the discriminant is negative, the trinomial $12x^2 + 5x + 2$ is always positive.

Based on the direction of the inequality and on the sign of the discriminant, the inequality admits no solution:

$$\nexists x \in \mathbb{R}$$

Esercizio. Solve the following inequality : $2x^2 - 3x > -2$

Solution: move -2 to the left-hand side:

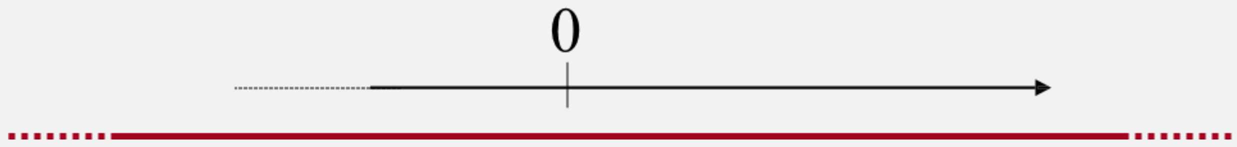
$$2x^2 - 3x + 2 > 0$$

Compute the discriminant $\Delta = b^2 - 4ac = 9 - 16 = -7 < 0$

If the discriminant is negative, the trinomial $2x^2 - 3x + 2$ is always positive.

Based on the direction of the inequality and on the sign of the discriminant, the inequality admits infinite solutions:

$$\forall x \in \mathbb{R}$$



Absolute Value Inequalities

Definition

The absolute value of x is equal to:

- x if x is greater or equal to zero
- $-x$ if x is less than zero

$$|x| \rightarrow \begin{cases} x \geq 0 \\ x \end{cases} \vee \begin{cases} x < 0 \\ -x \end{cases}$$

Absolute Value Inequalities

Inequalities with a single absolute value and a polynomial at the right-hand side

- $|A| > B \rightarrow \begin{cases} A \geq 0 \\ A > B \end{cases} \vee \begin{cases} A < 0 \\ -A > B \end{cases}$

- $|A| \geq B \rightarrow \begin{cases} A \geq 0 \\ A \geq B \end{cases} \vee \begin{cases} A < 0 \\ -A \geq B \end{cases}$

- $|A| < B \rightarrow \begin{cases} A \geq 0 \\ A < B \end{cases} \vee \begin{cases} A < 0 \\ -A < B \end{cases}$

- $|A| \leq B \rightarrow \begin{cases} A \geq 0 \\ A \leq B \end{cases} \vee \begin{cases} A < 0 \\ -A \leq B \end{cases}$

Absolute Value Inequalities

Inequalities with a single absolute value with a positive number n at the right-hand side

- $|A| > n \rightarrow A < -n \vee A > n$
- $|A| \geq n \rightarrow A \leq -n \vee A \geq n$
- $|A| < n \rightarrow \begin{cases} A < n \\ A > -n \end{cases}$
- $|A| \leq n \rightarrow \begin{cases} A \leq n \\ A \geq -n \end{cases}$

Inequalities with a single absolute value with a negative number $-n$ at the right-hand side

- $|A| > -n \rightarrow \forall x \in \mathbb{R}$
- $|A| \geq -n \rightarrow \forall x \in \mathbb{R}$
- $|A| < -n \rightarrow$ nessuna soluzione
- $|A| \leq -n \rightarrow$ nessuna soluzione

Inequalities with a single absolute value with zero at the right-hand side

- $|A| > 0 \rightarrow A \neq 0$
- $|A| \geq 0 \rightarrow \forall x \in \mathbb{R}$
- $|A| < 0 \rightarrow$ nessuna soluzione
- $|A| \leq 0 \rightarrow A = 0$

Absolute Value Inequalities

Inequalities with two or more absolute values

$$|A| + |B| \lessgtr C \quad \rightarrow \text{the signs of } A \text{ and } B \text{ are analysed}$$

- One solves the inequalities $A > 0$ and $B > 0$ (study of the signs) and, given e.g. $x > a$ and $x > b$ their solutions, they are represented on the following graph:

- Thanks to the graph observation, the following systems are obtained:

$$\text{I} \begin{cases} x < a \\ -A - B \gtrless C \end{cases} \quad \vee \quad \text{II} \begin{cases} a \leq x \leq b \\ A - B \gtrless C \end{cases} \quad \vee \quad \text{III} \begin{cases} x > b \\ A + B \gtrless C \end{cases}$$

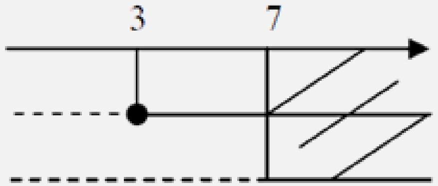
Exercise. Solve the following inequality: $|x-3| < 4$

From the definition of absolute value, the inequality is equivalent to the union of the systems:

$$\begin{cases} x-3 \geq 0 \\ x-3 > 4 \end{cases} \vee \begin{cases} x-3 < 0 \\ x-3 < -4 \end{cases}$$

One solves the system (a):

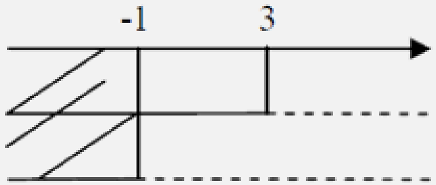
$$(a)\begin{cases} x \geq 3 \\ x > 7 \end{cases}$$



$x > 7$

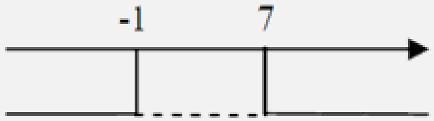
One solves the system (b):

$$(b)\begin{cases} x < 3 \\ x < -1 \end{cases}$$



$x < -1$

One combines the solutions:



This inequality could also be solved by squaring both the sides of the inequality itself:

$$|x-3| > 4 \Leftrightarrow (x-3)^2 > 16 \Leftrightarrow x^2 + 9 - 6x - 16 > 0 \Leftrightarrow x^2 - 6x - 7 > 0 \Leftrightarrow x < -1 \text{ e } x > 7$$

Irrational inequalities

Irrational inequalities with a square root and a polynomial at the right-hand side

- $\sqrt{A} > B \rightarrow \begin{cases} A \geq 0 \\ B < 0 \end{cases} \vee \begin{cases} B \geq 0 \\ A > B^2 \end{cases}$

- $\sqrt{A} \geq B \rightarrow \begin{cases} A \geq 0 \\ B < 0 \end{cases} \vee \begin{cases} B \geq 0 \\ A \geq B^2 \end{cases}$

- $\sqrt{A} < B \rightarrow \begin{cases} A \geq 0 \\ B > 0 \\ A < B^2 \end{cases}$

- $\sqrt{A} \leq B \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A \leq B^2 \end{cases}$

Irrational inequalities

Irrational inequalities with a positive number n at the right-hand side

- $\sqrt{A} > n \rightarrow A > n^2$
- $\sqrt{A} \geq n \rightarrow A \geq n^2$
- $\sqrt{A} < n \rightarrow \begin{cases} A \geq 0 \\ A < n^2 \end{cases}$
- $\sqrt{A} \leq n \rightarrow \begin{cases} A \geq 0 \\ A \leq n^2 \end{cases}$

Irrational inequalities with a negative number $-n$ at the right-hand side

- $\sqrt{A} > -n \rightarrow A \geq 0$
- $\sqrt{A} \geq -n \rightarrow A \geq 0$
- $\sqrt{A} < -n \rightarrow$ nessuna soluzione
- $\sqrt{A} \leq -n \rightarrow$ nessuna soluzione

Irrational inequalities with zero at the right-hand side

- $\sqrt{A} > 0 \rightarrow A > 0$
- $\sqrt{A} \geq 0 \rightarrow A \geq 0$
- $\sqrt{A} < 0 \rightarrow$ nessuna soluzione
- $\sqrt{A} \leq 0 \rightarrow A = 0$

Irrational inequalities

Irrational inequalities with two square roots (or in general with even index)

- $\sqrt{A} > \sqrt{B} \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A > B \end{cases}$

- $\sqrt{A} \geq \sqrt{B} \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A \geq B \end{cases}$

- $\sqrt{A} < \sqrt{B} \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A < B \end{cases}$

- $\sqrt{A} \leq \sqrt{B} \rightarrow \begin{cases} A \geq 0 \\ B \geq 0 \\ A \leq B \end{cases}$

To solve an inequality with two square roots, it is enough to isolate the roots on both sides and solve the system formed by the radicands set greater than or equal to zero and by the inequality obtained by squaring both sides.

Irrational inequalities

Irrational inequalities with cube roots

- With one cube root: $\sqrt[3]{A} \gtrless B \ \rightarrow A \gtrless B^3$

- With two cube roots: $\sqrt[3]{A} \lesseqgtr \sqrt[3]{B} \ \rightarrow A \lesseqgtr B$

To solve an inequality with cube roots, just isolate the root(s) and cube both sides

Exercise. Solve the following inequality: $x + 1 > \sqrt{x^2 + x - 2}$

One keeps the root on the left-hand side:

$$\sqrt{x^2 + x - 2} < x + 1$$

$$\sqrt{A} < B \rightarrow \begin{cases} A \geq 0 \\ B > 0 \\ A < B^2 \end{cases}$$

$$\sqrt{x^2 + x - 2} < x + 1 \quad \Rightarrow \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \begin{cases} x^2 + x - 2 \geq 0 \\ x + 1 > 0 \\ x^2 + x - 2 < (x+1)^2 \end{cases}$$

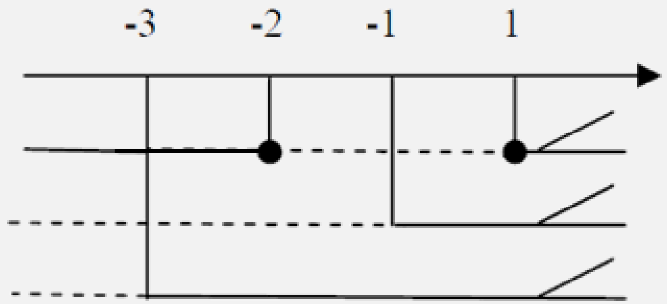
$(a)\ x^2 + x - 2 \geq 0$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2; 1 \quad \Rightarrow \quad x \leq -2 \quad e \quad x \geq 1$$

$(b)\ x + 1 > 0 \quad \Rightarrow \quad x > -1$

$(c)\ \cancel{x^2} + x - 2 < \cancel{x^2} + 2x + 1 \quad \Rightarrow \quad x + 3 > 0 \Rightarrow x > -3$

$$\begin{cases} x \leq -2; x \geq 1 \\ x > -1 \\ x > -3 \end{cases}$$



R: $x > 1$

REMEMBER

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $(x+y)(x-y) = x^2 - y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + 3y^3$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - 3y^3$

Remember: LOGARITHMS

the logarithm of a number is the exponent x to be given to the base a to obtain the argument b → $a^x = b$

$$log_a(b) = x$$

a = base → must be $a > 0$ and $a \neq 1$
b = argument → must be $b > 0$
x = logarithm of b in base a → is $\in \mathbb{R}$

Proprietà:

$log_a(a) = 1$	$log_a(1) = 0$	$a^x > 0$

Some theorems:

$log_a(b \cdot c) = log_a(b) + log_a(c)$	$log_2(3 \cdot x) = log_2(3) + log_2(x)$
$log_a\left(\frac{b}{c}\right) = log_a(b) - log_a(c)$	$log_2\left(\frac{x}{3}\right) = log_2(x) - log_2(3)$
$log_a(b)^c = c\ log_a(b)$	$log_2(x)^3 = 3\ log_2(x)$