- **FUNCTIONS**

Real function:

Let $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$.

A real function f from A (domain) to B (codomain), with real variable, is a correspondence that assigns to each element $x \in A$ one and only one element $y \in B$. It is denoted by:

$$f : A \to B$$

$$f : x \in A \to y \in \mathrm{B}$$

- A is called **domain** of f, also denoted with $dom(f)$, Df

- B is called **codomain** of f ,

The rule that associates to the **independent variable** x its **image** $f(x)$ defines the function.

- The image of a function is the set of all possible values $f(x)$ associated with at least one element of the domain:

$$im(f) = \{y \in B : y = f(x), x \in A\}$$

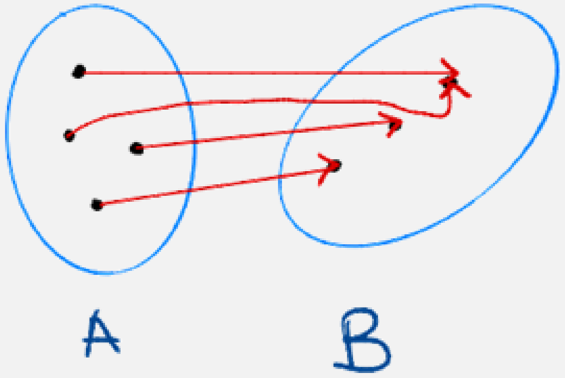
Uniqueness of the Image

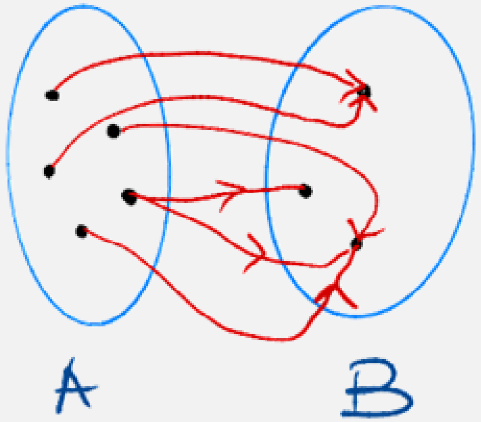
For a correspondence to be a function, it must assign uniquely to each $x \in dom(f)$ a value $f(x)$.

It is not necessary that different values of x correspond to different values of $f(x)$: it may happen that for $x_1, x_2 \in dom(f)$, with $x_1 \neq x_2$ one has

$$f(x_1) = f(x_2)$$

$$\forall x \in A, \quad \exists! y \in B : y = f(x).$$





The definition permits that the same y corresponds to different x values

Analytical expression:

A function is defined by providing its analytical expression, that is, a set of well-defined mathematical operations which, when applied in a given order, allow us to compute the corresponding value $y = f(x)$ for each assigned value of x

$$y = x + 1 \Rightarrow \text{ if } \quad x = 1, y = 2; \text{ if } x = 2, y = 3; \ldots$$

$$y = x^3 + x^2 \Rightarrow \text{ if } \quad x = 1, y = 2; \text{ se } x = 2, y = 12; \ldots$$

Note: In the case of a real function of a real variable $f : dom(f) \to \mathbb{R}$, we will always consider $\mathbb{R}$ as the codomain.

Functions can be injective, surjective, or bijective:

- $f\colon A \to B$ is **injective** if

$$\forall a_1, a_2 \in A, \qquad a_1 \neq a_2 \;\Rightarrow f(a_1) \neq f(a_2)$$

$$\Updownarrow$$

$$\forall a_1, a_2 \in A, \qquad f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

one takes two different elements and their images are different; viceversa is valid

- $f\colon A \to B$ is **surjective** if $im(f) = B$

all the elements of B are taken in the function; this does not imply, however, that it is injective, since different a_1 and a_2 can have the same image:

$$\forall y \in B \; \exists x \in A : f(x) = y$$

- $f\colon A \to B$ is **bijective** if it is both injective and surjective.

In this case, the vice versa is valid, and an inverse function exists.

The graph of a function $f : D \to \mathbb{R}$

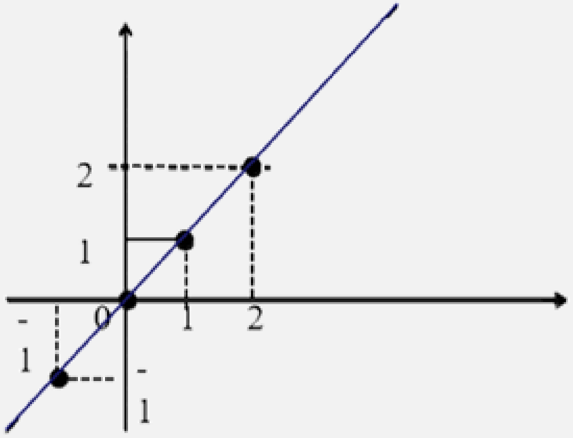
is the set of all points in the Cartesian plane Oxy with coordinates (x, y) where x belongs to the domain D and $y = f(x)$

$$G_f = \{(x, y) : x \in D, y = f(x)\}$$

$f : x \in R \to x \in R$

$f(x) = x$

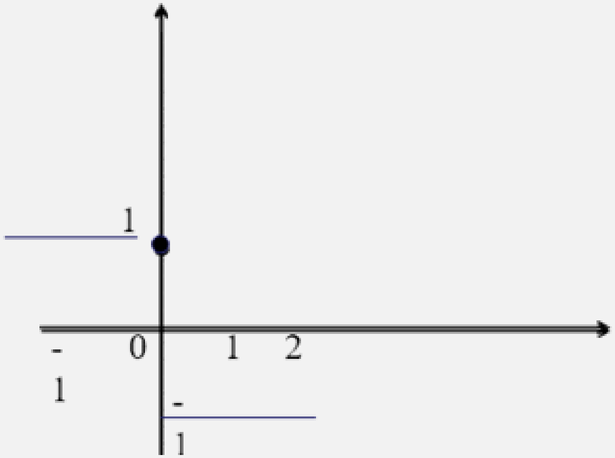
x	f(x)
-1	-1
0	0
1/2	1/2
1	1
2	2
…	…



The graph is the line passing through the identified points

$$f : x \in R \to \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

x	f(x)
-2	-1
-1	-1
0	1
1/2	1
1	1
…	…



Piecewise-defined function

Operations Between Functions

Functions can be combined through arithmetic operations.

Let $f : A \to \mathbb{R}, \quad g : A \to \mathbb{R}$ be two real functions with the same domain:

- $(f + g)x = f(x) + g(x)$
- $(f - g)x = f(x) - g(x)$
- $(f \cdot g)x = f(x) \cdot g(x)$
- $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, con\ g(x) \neq 0$

Operations Between Functions

COMPOSITION OF FUNCTIONS

Let $f : A \to B, \quad g: B \to C$, **the composition of $\boldsymbol{f}$ $\boldsymbol{and}$ $\boldsymbol{g}$** is the function from A to C denoted by:

$$g \circ f : A \to C$$

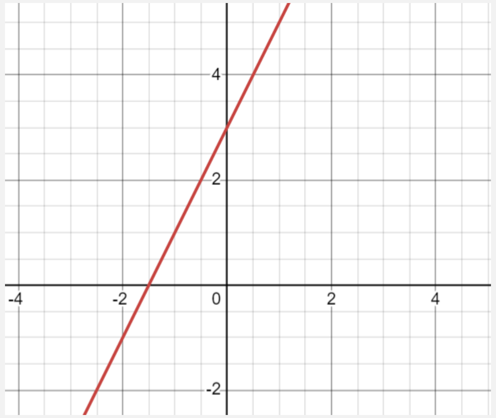
Defined as:

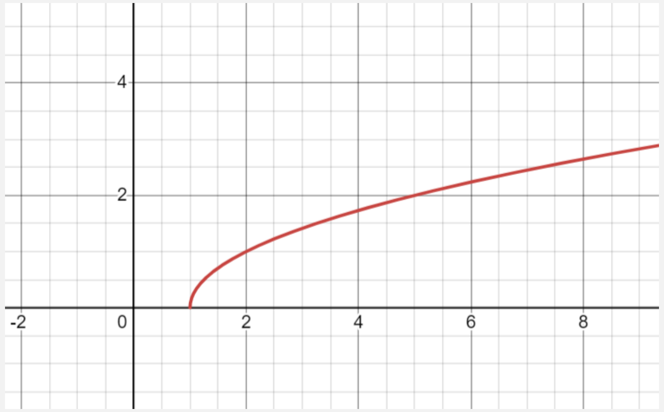
$$(g \circ f)(x) = g\big(f(x)\big)$$

Note. First compute f then apply g

Example. $f(x) = 2x + 3$ e $g(x) = \sqrt{x-1}$

$(g \circ f) = g\big(f(x)\big) = g(2x+3)$
$= \sqrt{2x-2} = \sqrt{2(x-1)}$





$(f \circ g) = f\big(g(x)\big) = f\big(\sqrt{x-1}\big)$
$= 2\sqrt{x-1} + 3$

The results are different: composition is not commutative.

Determining the Domain

Determining the domain of a function means imposing conditions that exclude values for which the function is not defined.

$$g(x) = \sqrt{x-1} \qquad x-1 \geq 0 \ \rightarrow x \geq 1$$

$$Df = \{x \in \mathbb{R} : x \geq 1\}$$

RULES:

- For even-order roots, the radicand must be ≥ 0
- For fractions, the denominator must be $\neq 0$:

  $f(x) = \frac{1-x}{x^2-3}$ → all the operations are allowed singularly, but it is necessary that

  $$x^2 - 3 \neq 0 \ \rightarrow \ x^2 \neq 3 \ \rightarrow x \neq \pm\sqrt{3}$$

  $$Df = \{x \in \mathbb{R} : x \neq \pm\sqrt{3}\}$$

- For logarithms, the argument must be > 0:

  $$f(x) = 2 - xlog(x+5) \rightarrow \ x+5 > 0 \ \rightarrow x > -5$$

  $$Df = \{x \in \mathbb{R} : x > -5\}$$

Domain and Image (or Codomain)

$$g(x) = \sqrt{x-1} \qquad x - 1 \geq 0 \ \rightarrow x \geq 1$$

$$Df = \{x \in \mathbb{R} : x \geq 1\}$$

$\boldsymbol{Df}$: set of values that x can assume (domain) (definition set)

The image of x through f is the corresponding value y;

$\boldsymbol{Cf}$ or $\mathrm{Im}(\boldsymbol{f})$: set of all values assumed by the function.

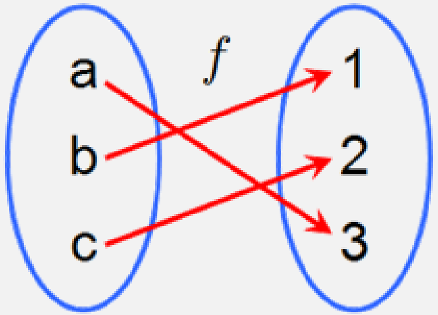
Inverse function

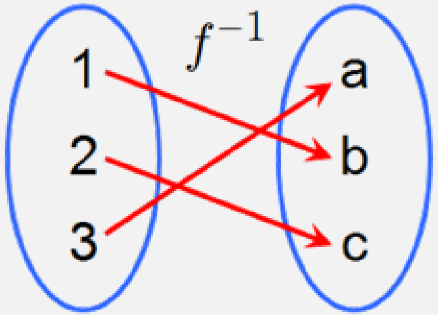
Given a bijective function:

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

the inverse function, denoted by f^{-1} is defined as:

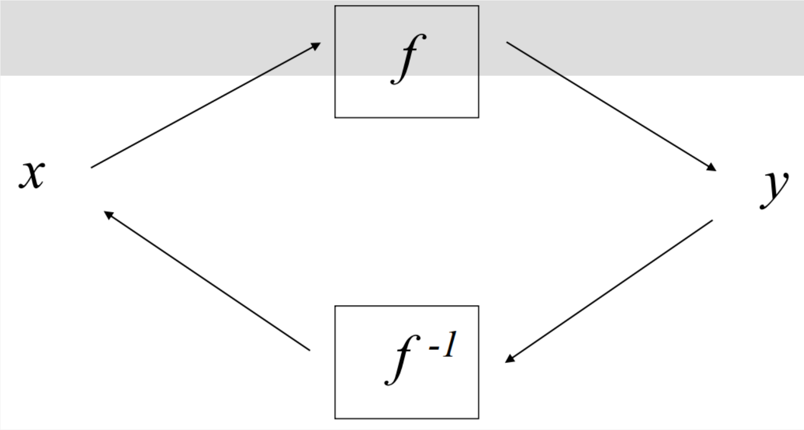
$$f^{-1}: \underset{y}{f(A)} \longrightarrow \underset{x}{A}$$





Such that $\forall y \in f(A)$ there exists a <u>unique</u> $x \in A$ with:

$$x = f^{-1}(y)$$



If one starts from x and apply first f and then f^{-1} , one returns to x.

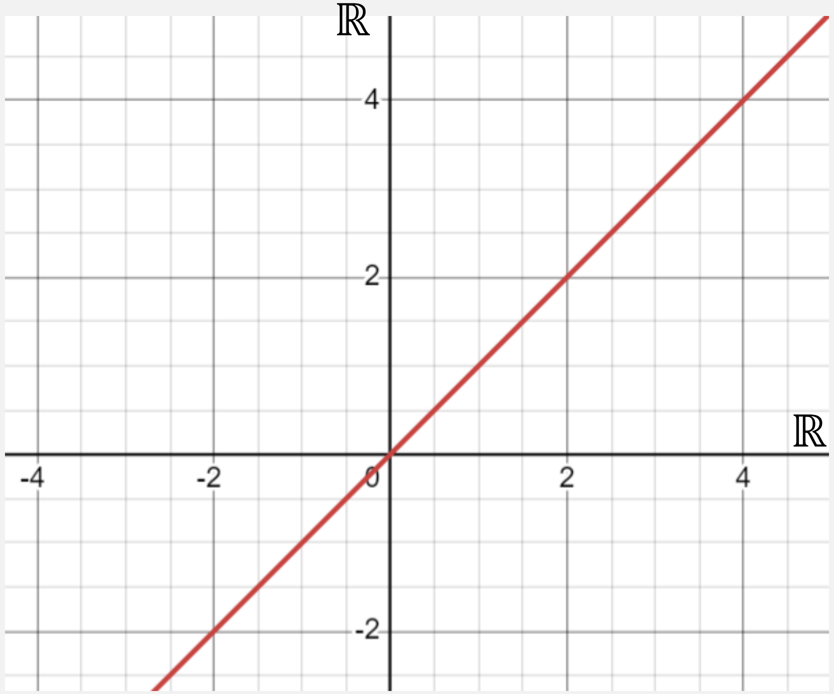
$$f^{-1}(f(x)) = x, \qquad \forall x \in A$$

$$(f^{-1} \circ f)(x) = x$$

$(f^{-1} \circ f)$ is called the **identity function** of A

$(f^{-1} \circ f)(a)$ <u>identity function</u> of the set A

$(f \circ f^{-1})(b)$ <u>identity function</u> of the set B



Bounded functions

Given a function:

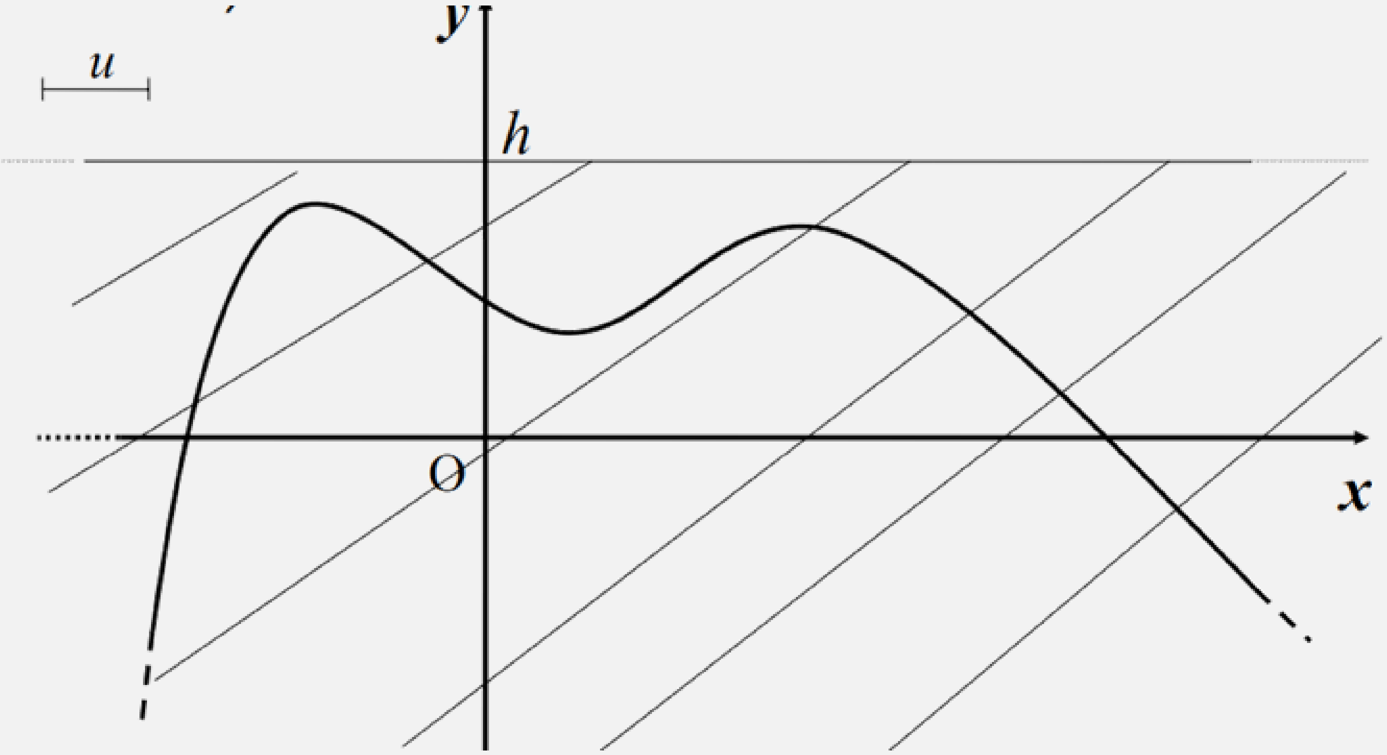
$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

- The function f is bounded above if its image $f(A)$ is bounded above, i.e. if $f(A)$ admits upper bounds.
- The function f is bounded below if its image $f(A)$ is bounded below, i.e. if $f(A)$ admits lower bounds.
- The function f is bounded if its image $f(A)$ is bounded both above and below.

Function Bounded Above

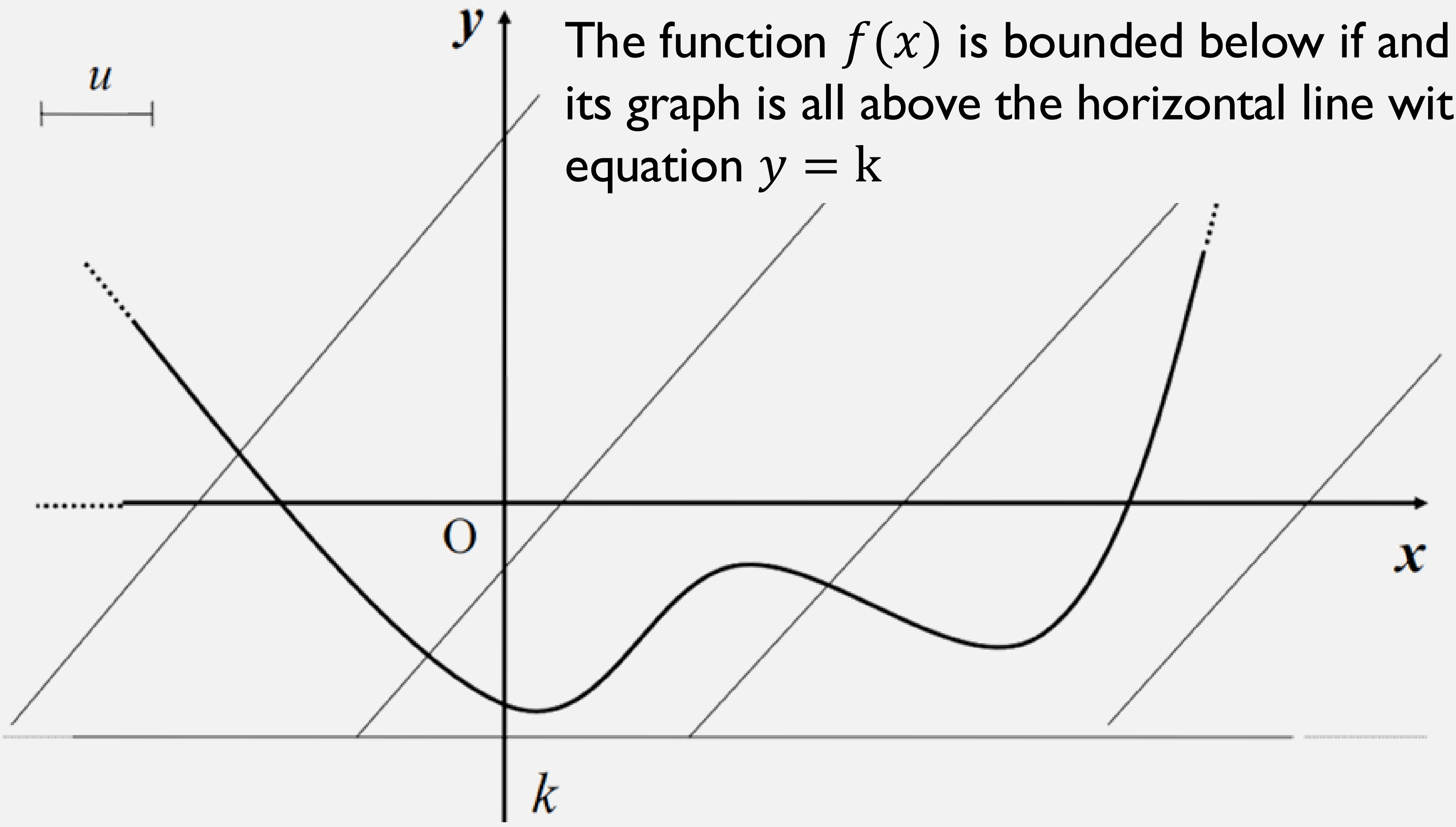
$$\textit{Function Bounded Above} \Leftrightarrow \exists\, h \in \mathbb{R} : f(x) \leq h, \forall x \in A$$

The function $f(x)$ is bounded above if and only if its graph is all under the horizontal line with equation $y = h$



Function Bounded Below

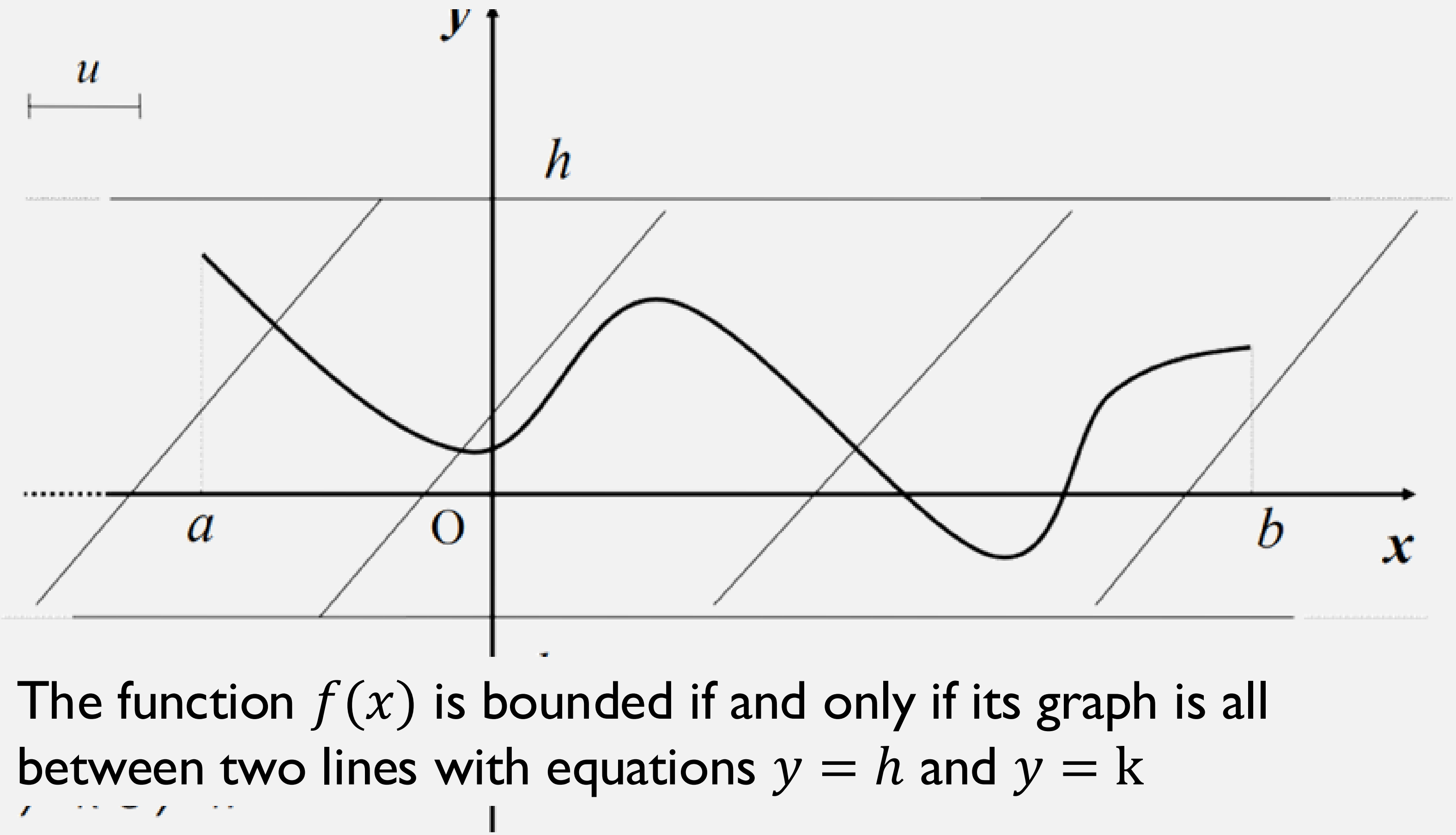
$$f \text{ bounded below} \Leftrightarrow \exists k \in \mathbb{R} : f(x) \geq k, \forall x \in A$$



The function $f(x)$ is bounded below if and only if its graph is all above the horizontal line with equation $y = \mathrm{k}$

Bounded Function

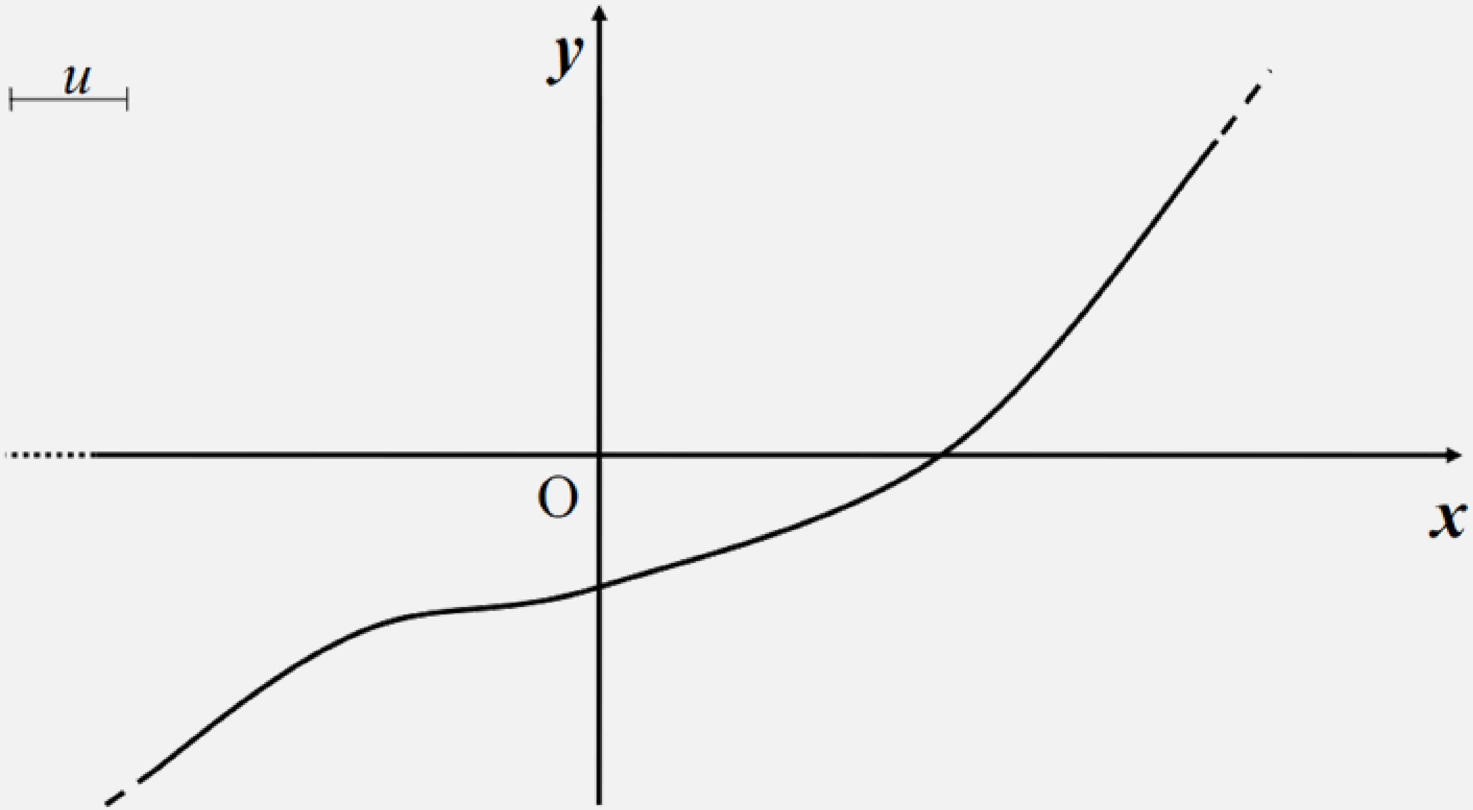
$$f \text{ bounded} \Leftrightarrow \exists h, k \in \mathbb{R} : k \leq f(x) \leq h, \forall x \in A$$



The function $f(x)$ is bounded if and only if its graph is all between two lines with equations $y = h$ and $y = \mathrm{k}$

Unbounded function

An unbounded function is a function whose image set admits neither upper bounds nor lower bounds.



Supremum

Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

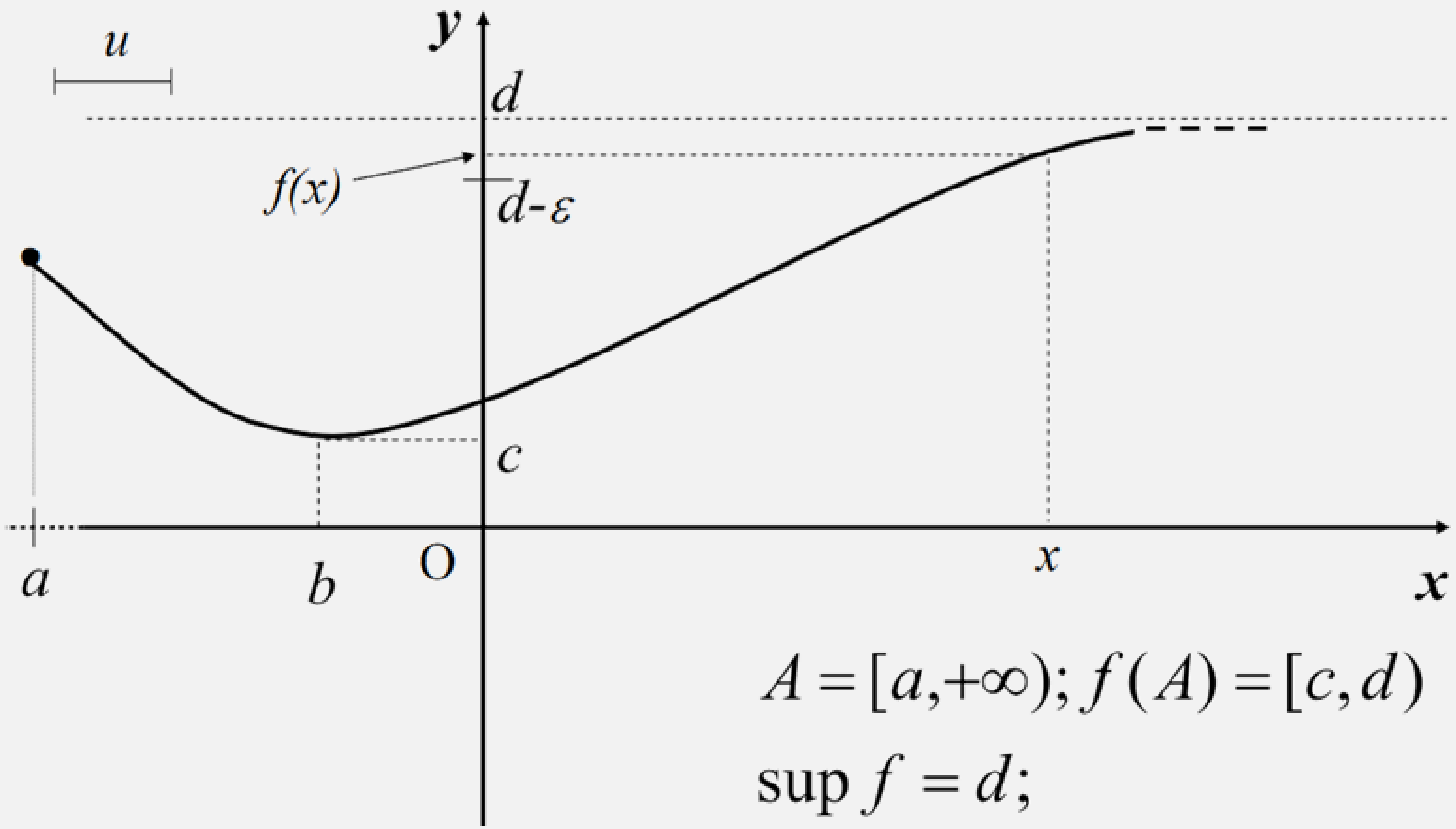
Be a function bounded above (⇒ its image $f(A)$ admits upper bounds).

The number M is called the **supremum** of f if it is the supremum of the image set $f(A)$:

$$M = \sup f \Leftrightarrow \begin{cases} \forall x \in A, f(x) \leq M \\ \forall \varepsilon > 0, \exists x \in A : M - \varepsilon < f(x) \end{cases}$$

If f is not bounded above, then: $\sup f = +\infty$

Exercise. From the graph, deduce domain, codomain, supremum



$$A = [a, +\infty);\ f(A) = [c, d)$$

$$\sup f = d;$$

Infimum

Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

be a function bounded below. (⇒ its image $f(A)$ admits lower bounds).

The number m is called the **infimum** of f if it is the infimum of the image set $f(A)$:

$$m = \inf f \Leftrightarrow \begin{cases} \forall x \in A, f(x) \geq m \\ \forall \varepsilon > 0, \exists x \in A : m + \varepsilon > f(x) \end{cases}$$

If f is not bounded below, then: sup$f$=-∞

Absolute Maximum

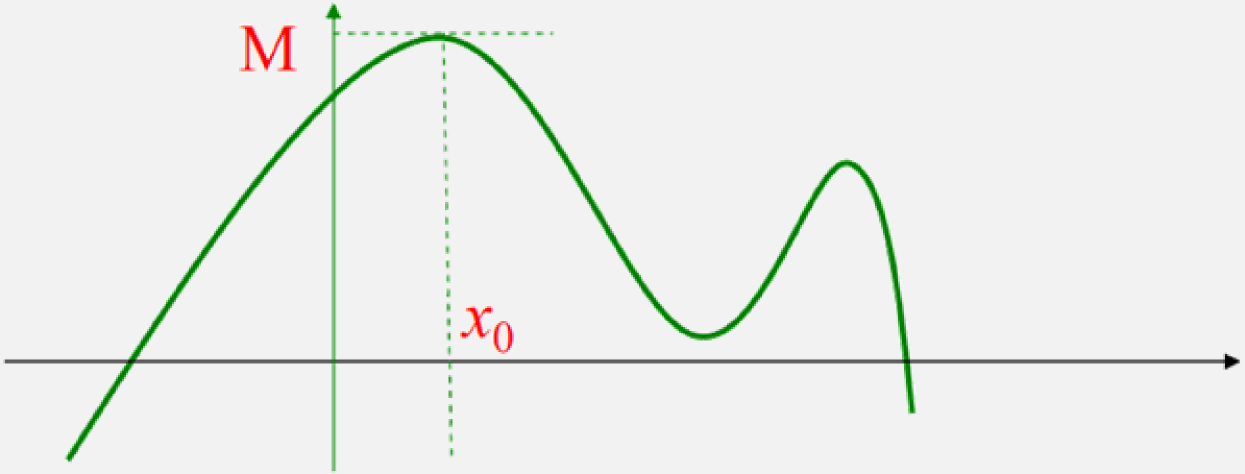
Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

A real number M is called the **absolute maximum** of f if M is a value belonging to the image of f and if it is the biggest value:

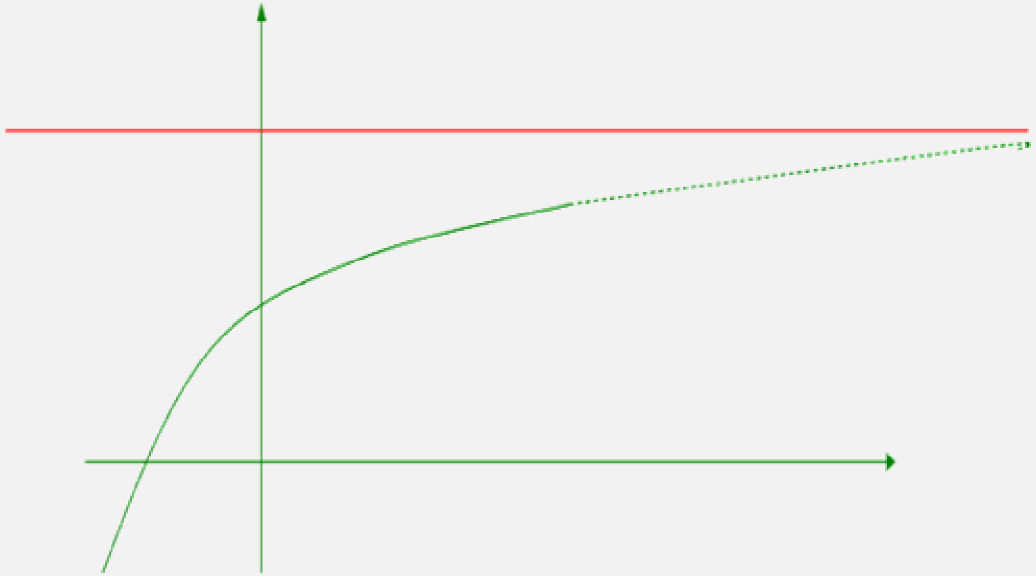
$$M = \max f \Leftrightarrow \begin{cases} \exists x_0 \in A : f(x_0) = M \\ \forall x \in A, f(x) \leq M \end{cases}$$

with x_0 called the **point of absolute maximum**



Properties of the Absolute Maximum

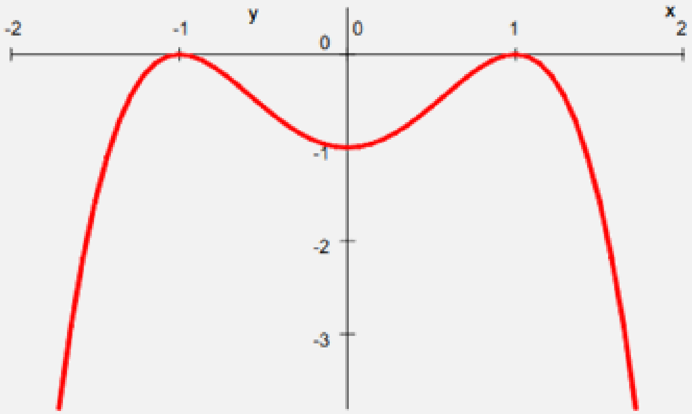
- The maximum of a function, if it exists, is the greatest value attained by the function.
- If a function admits an absolute maximum, then it is bounded above.
- If a function is bounded above, it admits a supremum, but not necessarily a maximum.
- A function may have more than one point of absolute maximum.



$$f(x) = -(x-1)^2(x+1)^2$$

$$M = 0$$

$$x_0 = 1; \quad x_1 = -1$$



Absolute Minimum

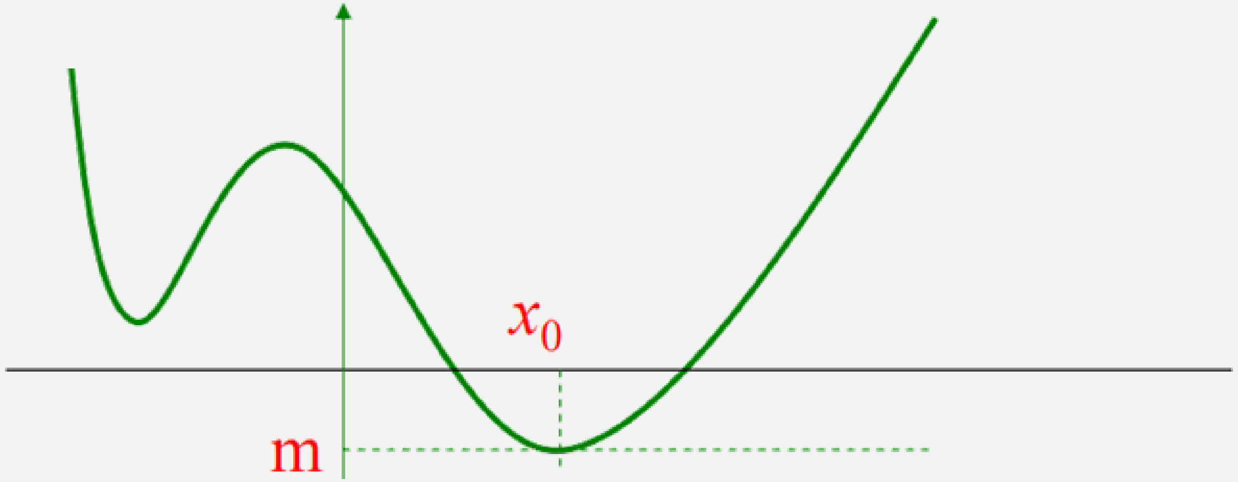
Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

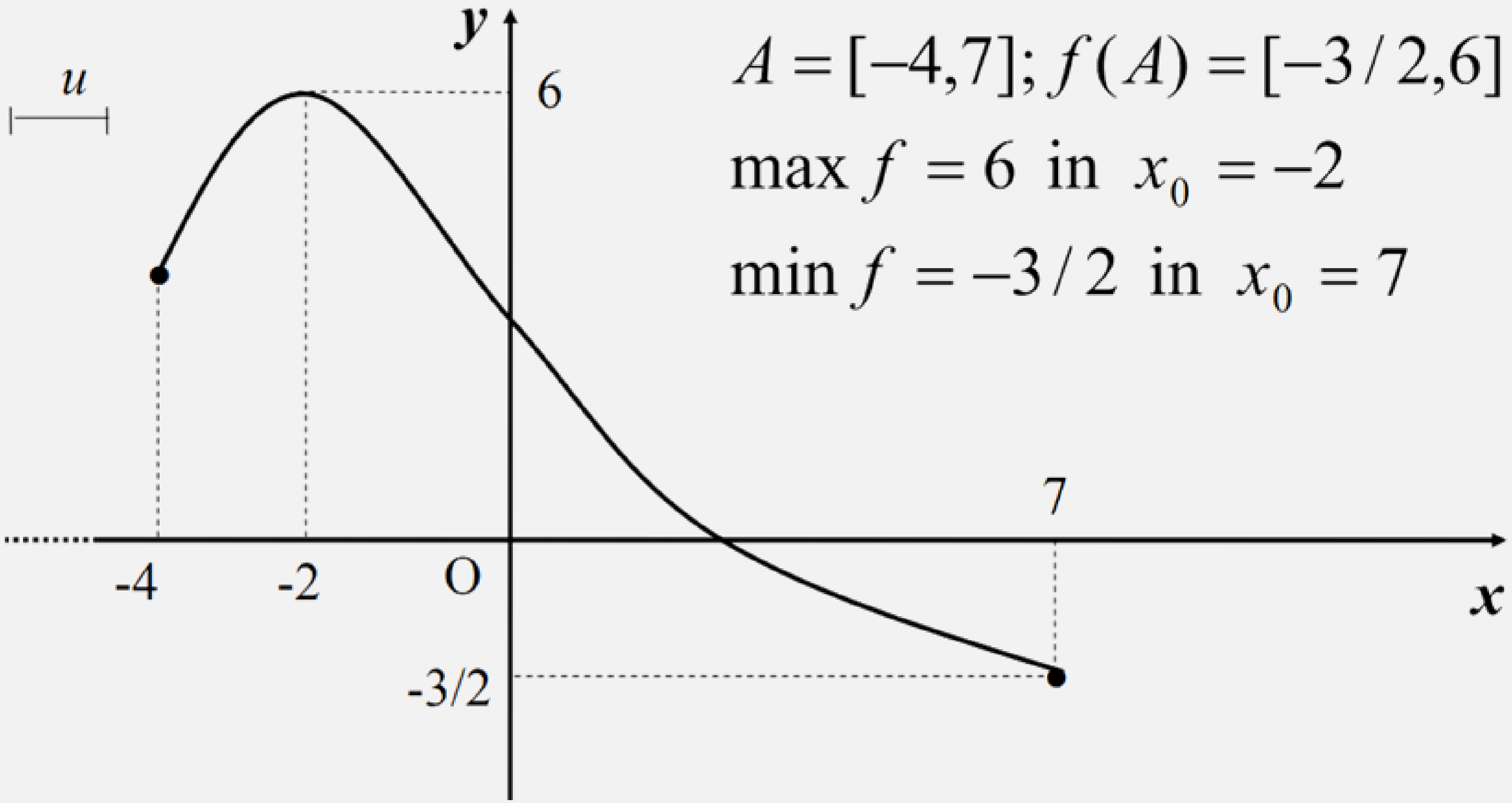
A real number m is called the absolute minimum of f if m is a value belonging to the image of f and if it is the smallest value:

$$m = \min f \Leftrightarrow \begin{cases} \exists x_0 \in A : f(x_0) = m \\ \forall x \in A, f(x) \geq m \end{cases}$$

with x_0 called the **point of absolute minimum.**



Exercise. From the graph, deduce domain, codomain, maximum and minimum.



$$A = [-4,7];\ f(A) = [-3/2, 6]$$

$$\max f = 6 \ \text{ in } \ x_0 = -2$$

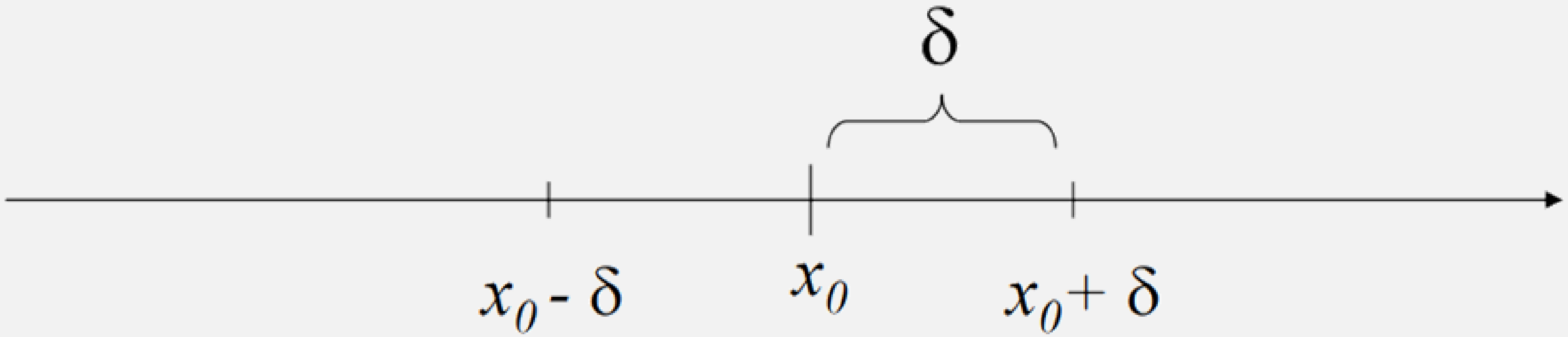
$$\min f = -3/2 \ \text{ in } \ x_0 = 7$$

Neighborhood

Fix a point x_0 on the real axis. A **neighborhood $\boldsymbol{I_{x0}}$** of the point x_0 is an open interval containing x_0.

If δ denotes the half-width of the interval, ther neighborhood is written as:

$$(x_0 - \delta, x_0 + \delta)$$



Local (Relative) Maximum

Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

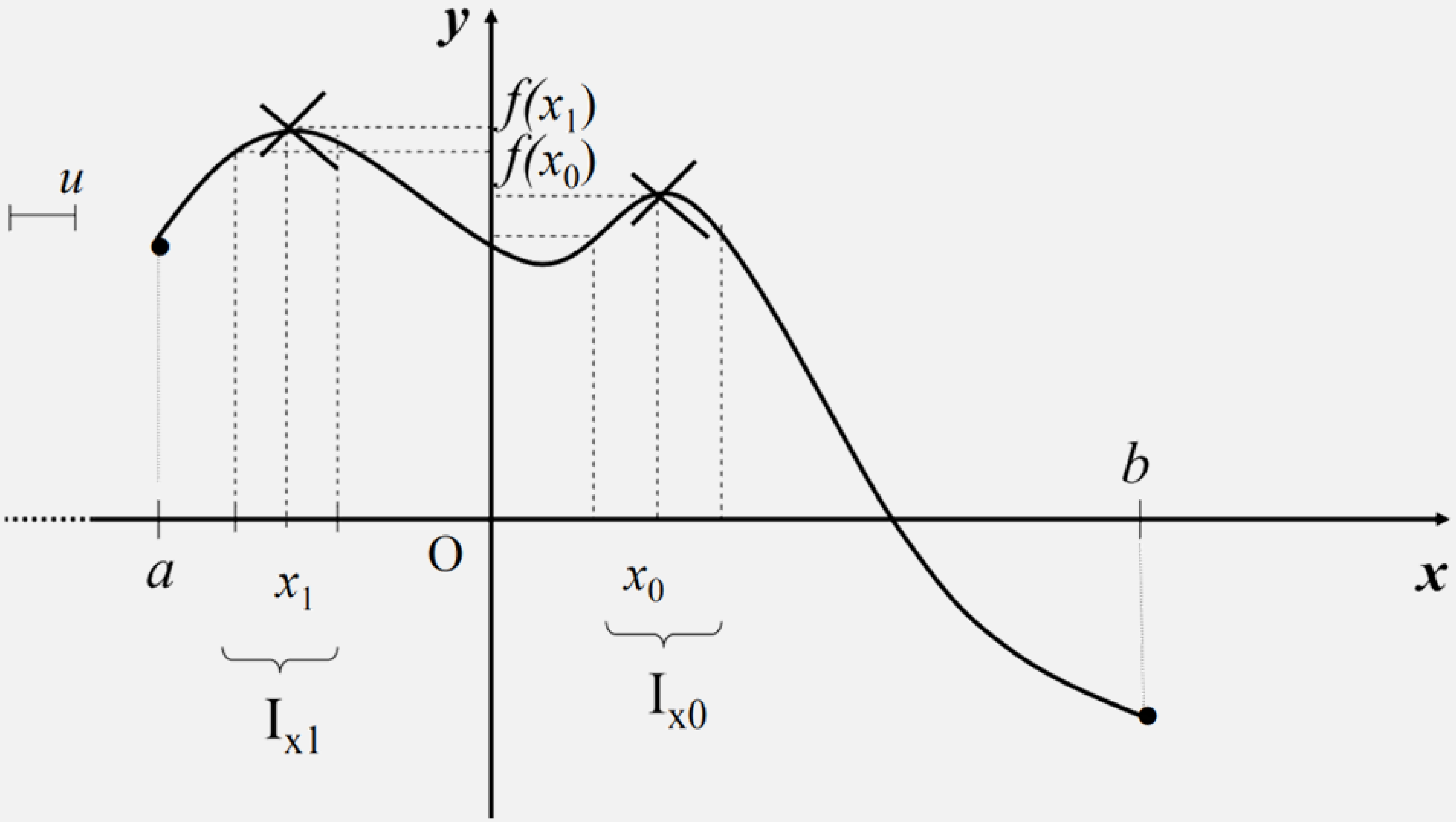
Given a point $x_0 \in \mathrm{A}$, the value $L = f(x_0)$ is a **local (relative) maximum if**:

$$\exists\, I_{x0} \text{ such that}$$

$$\forall x \in I_{x0} \cap A, f(x) \leq L$$

The point x_0 is called **point of local maximum**

Local Maximum (Graphical Representation)



Local (Relative) Minimum

Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

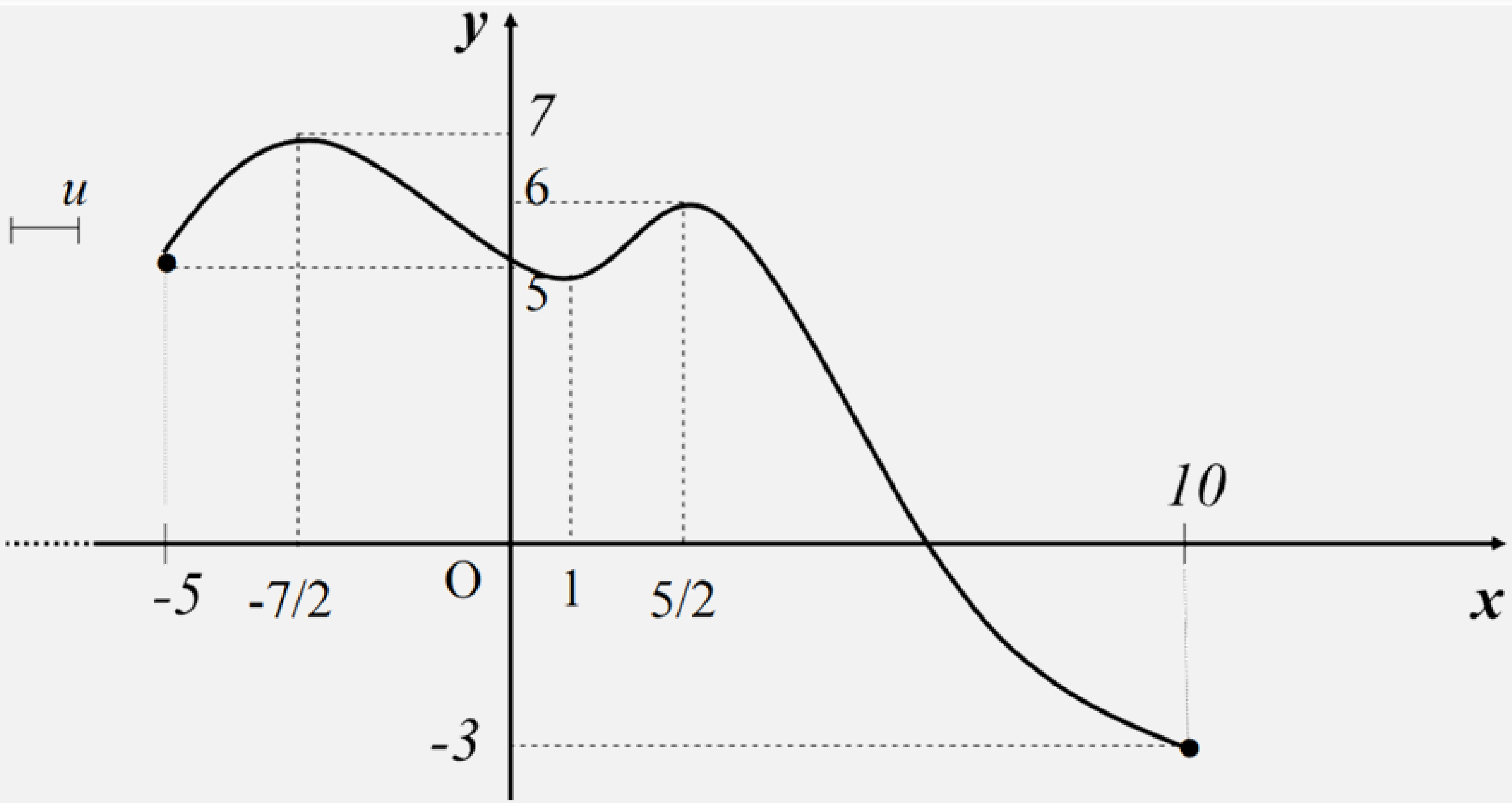
Given a point $x_0 \in \mathrm{A}$, the value $l = f(x_0)$ is a **local (relative) minimum** if:

$$\exists\, I_{x0} \; such\ that$$

$$\forall x \in I_{x0} \cap A, f(x) \geq l$$

The point x_0 is called **point of local minimum**

Exercise. From the graph, deduce domain, codomain, maximum (rel and abs), minimum (abs and rel), supremum, infimum, the value of $f(10)$, and the value of x such that $f(x) = 5$.



Monotone Functions: Increasing

Let

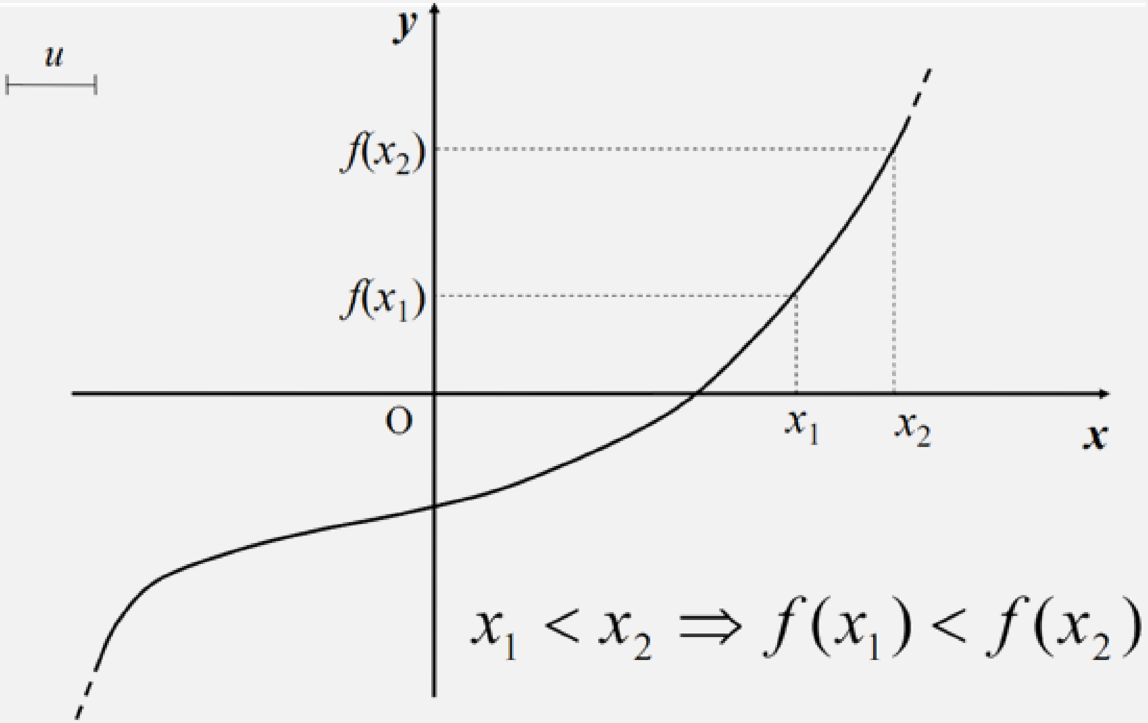
$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$

f is **strictly increasing** in A if:

$$\forall x_1, x_2 \in A : x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

f is **increasing** in A if:

$$\forall x_1, x_2 \in A : x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$



Monotone Functions: Decreasing

Let
$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$

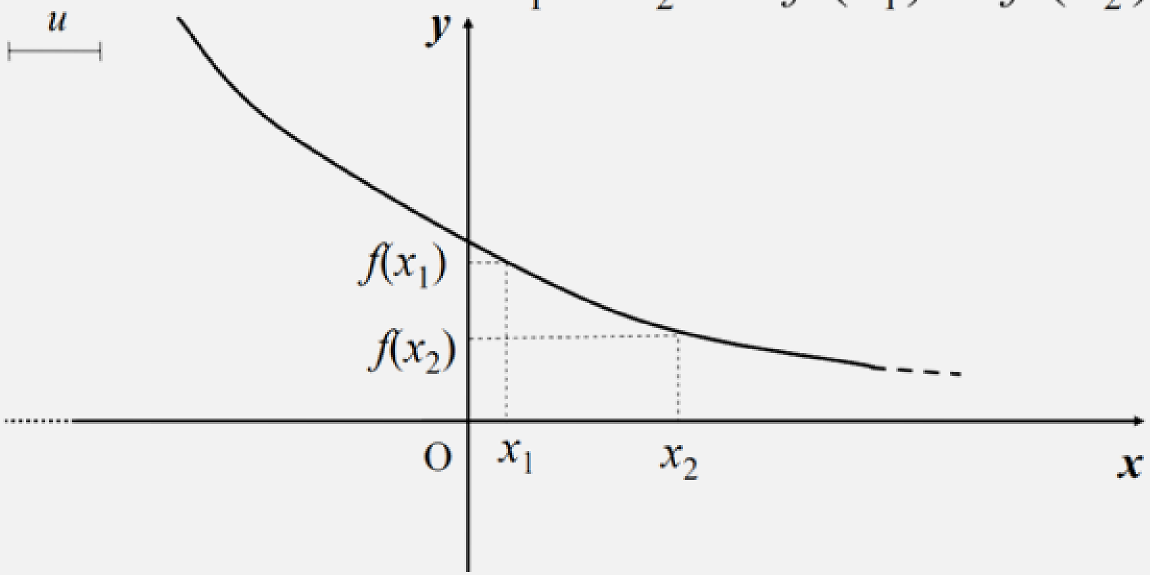
f is **strictly decreasing** in A if:

$$\forall x_1, x_2 \in A : x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

f is **decreasing** in A if:

$$\forall x_1, x_2 \in A : x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



Monotone Functions

Let

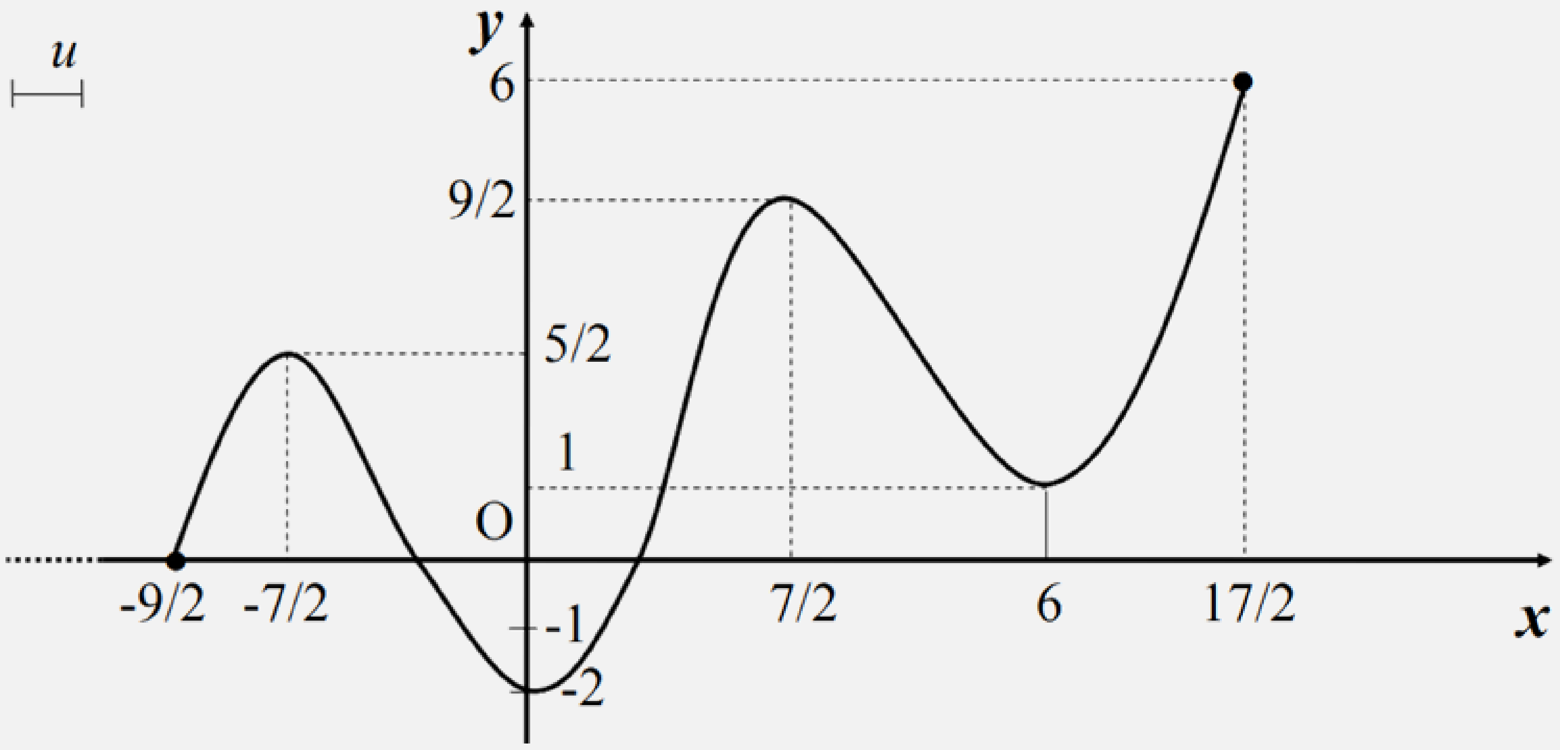
$$f : A \to B, \qquad con\, A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

A function f is called monotone if it is:

- strictly increasing,
- increasing,
- strictly decreasing, or
- decreasing.

Warning: The notion of monotonicity is always associated with the interval of the domain on which the function exhibits that behavior.

Example. From the graph of f deduce domain, codomain and monotonicity



Positive/Negative Function and Zeros

Let

$$f : A \to B, \qquad \text{with } A, B \subseteq \mathbb{R}, \qquad A, B \neq \emptyset$$

The function f is said to be **positive** on A if:

$$\forall x \in A \Rightarrow f(x) > 0$$

The function f is said to be **negative** on A if:

$$\forall x \in A \Rightarrow f(x) < 0$$

A point x_0 is called a **zero** of f in A if:

$$f(x_0) = 0$$

Example. From the graph of f deduce domain, codomain and sign