ELEMENTARY FUNCTIONS

- **Linear function**
- **Polynomial function**
- **Absolute value function**
- **Power function**
- **Exponential function**
- **Logarithmic function**
- **Trigonometric functions**

ELEMENTARY FUNCTIONS

- **Linear function**

Linear function

A **linear function** is defined as a function whose general form is:

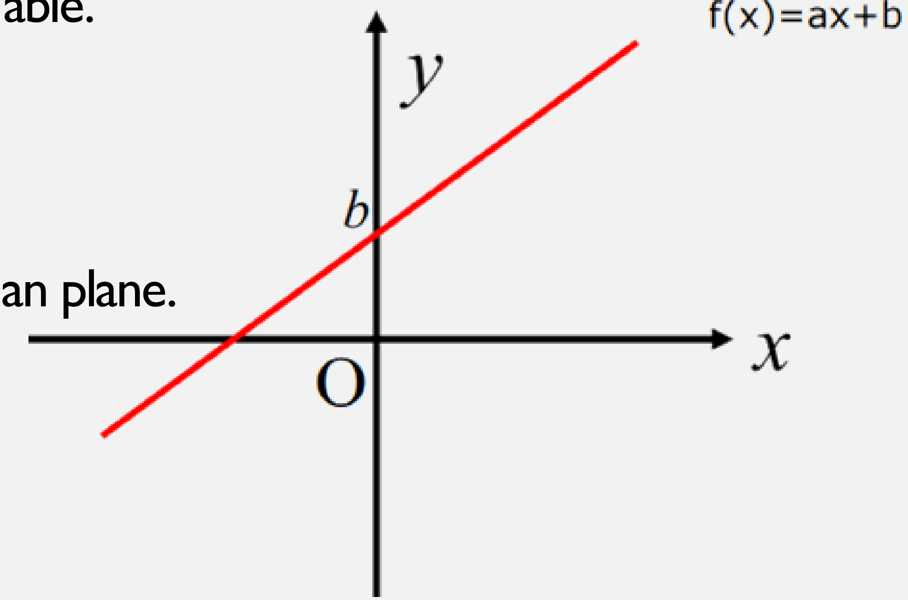
$$f(x) = ax + b, \qquad a, b \in \mathbb{R}$$

$$f : x \in \mathbb{R} \to (ax + b) \in \mathbb{R}$$

The constant a is called the slope (or angular coefficient) of the function and represents the rate of change of the function with respect to the independent variable.

The constant b is called the y-intercept (or constant term).

The graph of any linear function is a straight line in the Cartesian plane.



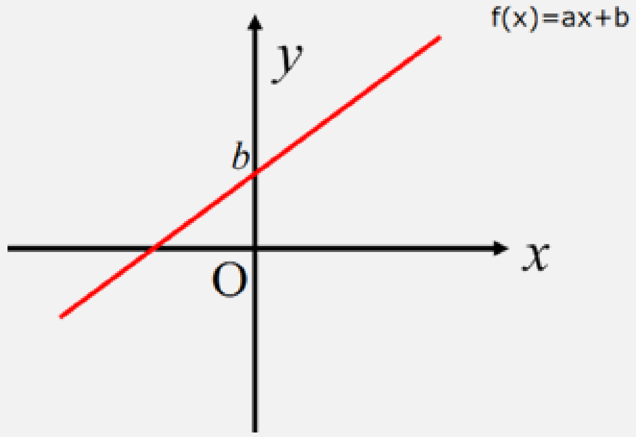
The constant term b

A linear function is characterized by two constants, a and b.

If $x = 0 \rightarrow\ y = b$, therefore the point $P(0, b)$ which lies on the vertical y-axis, belongs to the graph of the function.

The constant term b is the ordinate of the intersection of the graph of the function with the y-axis and is called the **INTERCEPT**.

If the independent variable is time and the function is written as $f(t) = at + b$, the coefficient $b = f(0)$ is called the INITIAL VALUE

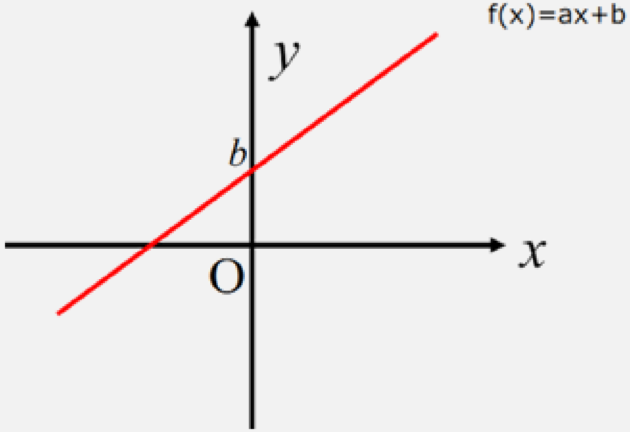


The slope a

Given the function $f(x) = ax + b$, the coefficient a represents the **variation of the ordinates** of two arbitrary points on the graph with respect to the corresponding **variation of the abscissas**:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- The coefficient a quantifies the **inclination** of the straight line with respect to the horizontal axis.
- The coefficient a is **constant**.

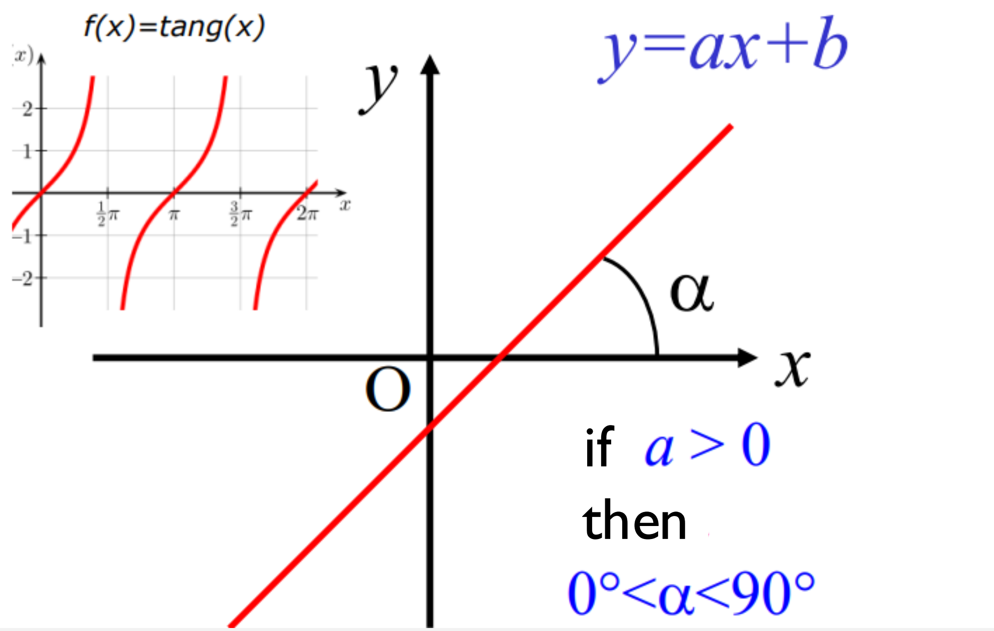


The slope a

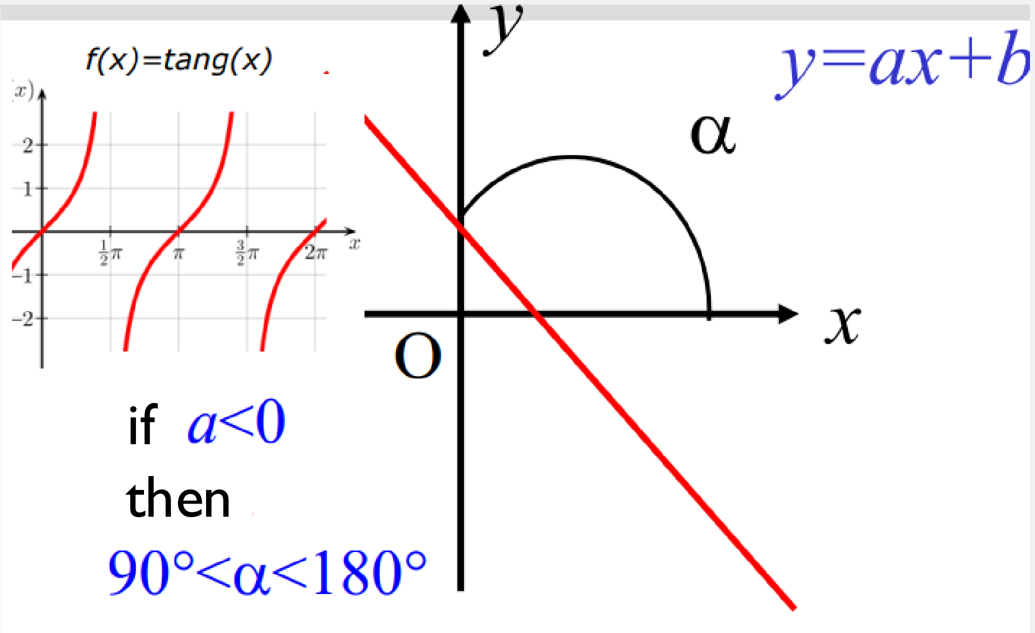
The slope a is given by:

$$a = \tan \alpha$$

where α is the angle formed by the line and the horizontal axis, measured counterclockwise



f strictly increasing in $\mathbb{R}$
$\sup(\mathrm{ax} + \mathrm{b}) = +\infty;\ \inf(\mathrm{ax} + \mathrm{b}) = -\infty$



f strictly decreasing in $\mathbb{R}$
$\sup(\mathrm{ax} + \mathrm{b}) = +\infty;\ \inf(\mathrm{ax} + \mathrm{b}) = -\infty$

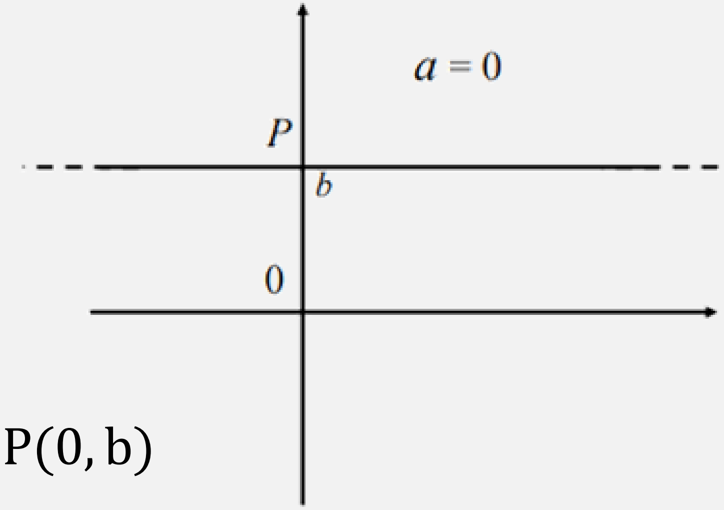
The slope a

The slope a is given by:

$$a = \tan \alpha$$

where α is the angle formed by the line and the horizontal axis, measured counterclockwise

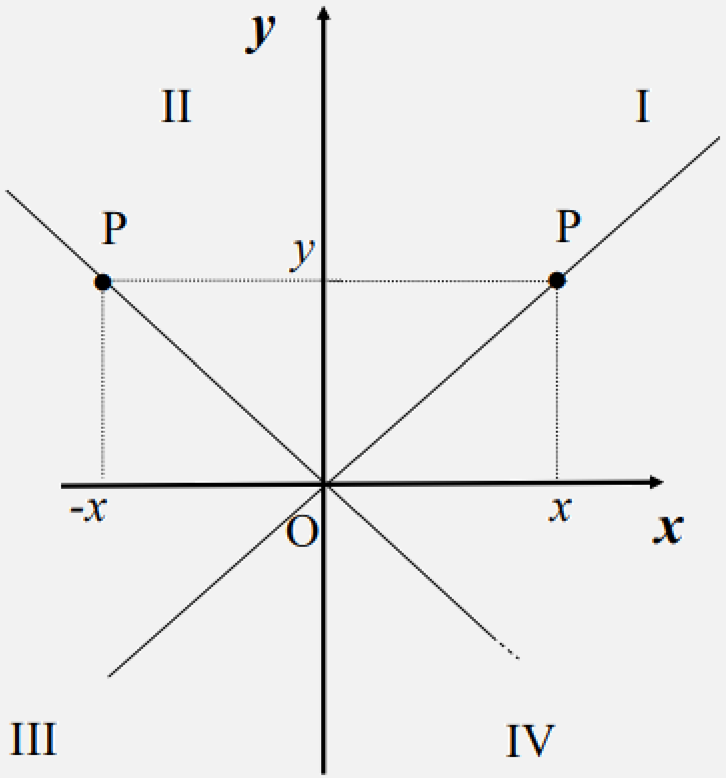
$$f(x) = b, \quad b \in R$$



f constant in $\mathbb{R}$ for $\mathrm{P}(0, \mathrm{b})$

Two particular linear functions are those whose graphs are the **angle bisectors**.

- All points lying on the bisector of the first and third quadrants have equal abscissa and ordinate.
- All points lying on the bisector of the second and fourth quadrants have opposite abscissa and ordinate.



$$\boldsymbol{f(x) = mx + q}$$

Bisector of I and III quadrants

$$f(x) = x$$

$$\Downarrow$$

$m = 1$ e $q = 0$

f strictly increasing

Bisector of II and IV quadrants

$$f(x) = -x$$

$$\Downarrow$$

$m = -1$ e $q = 0$

f strictly decreasing

Graphical solution of a first-degree equation

Solving graphically a first-degree equation of the form

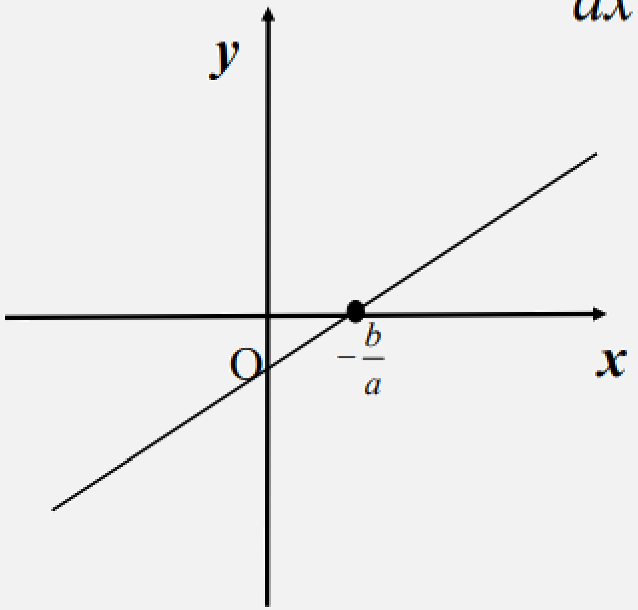
$$ax + b = 0 \text{ con } a, b \in \mathbb{R}$$

means determining the points of intersection between the x-axis and the graph of the linear function

$$f(x) = ax + b$$

which is represented by a straight line with slope a and y-intercept b

$a > 0$



$$ax + b = 0 \Leftrightarrow x = -\frac{b}{a}$$

Graphical solution of a first-degree inequality

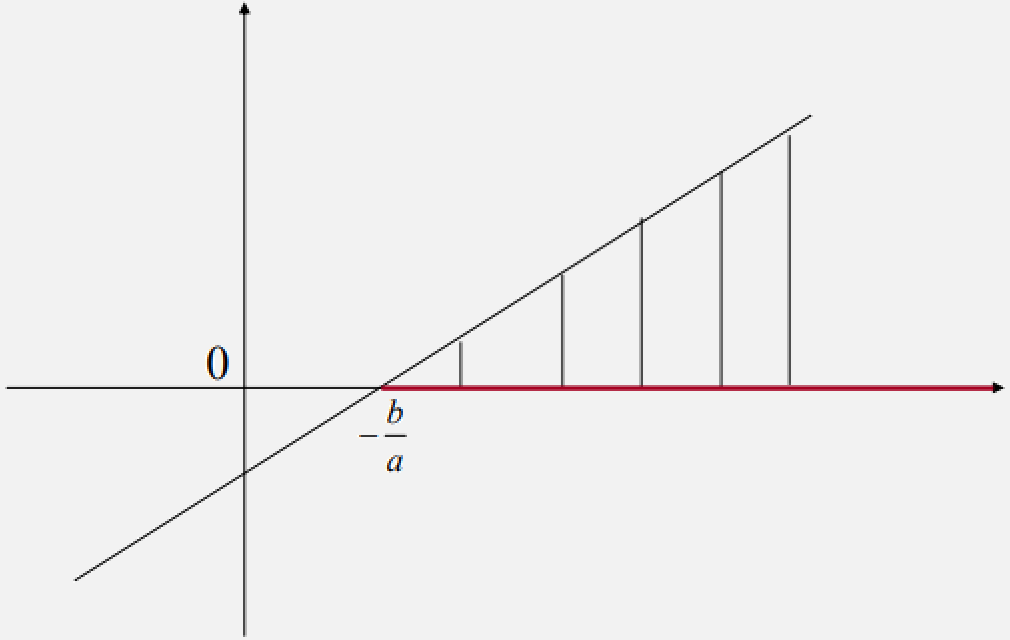
Solving graphically a first-degree inequality of the form

$$ax + b > 0 \text{ con } a, b \in \mathbb{R}$$

means determining the abscissas of the points of the graph of the linear function $f(x) = ax + b$ that lie above the x-axis

$a > 0$

$$ax + b > 0 \Leftrightarrow x > -\frac{b}{a}$$



- **ELEMENTARY FUNCTIONS**
  - **Absolute value function**

Absolute value function

The **absolute value function** is defined by:

$$f(x) = |x|$$

$$f : x \in \mathbb{R} \to |x| \in [0, +\infty)$$

where

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

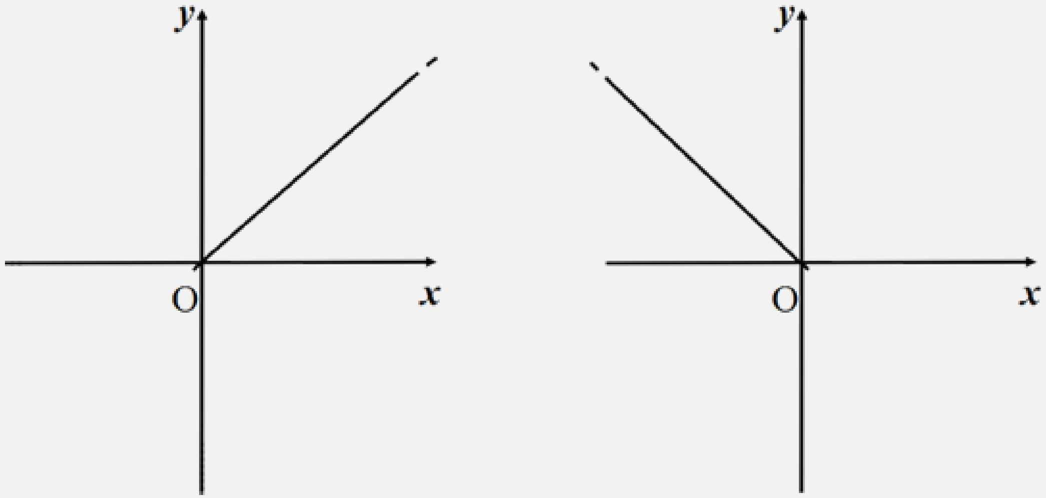
Its graph consists of two half-lines passing through the origin.

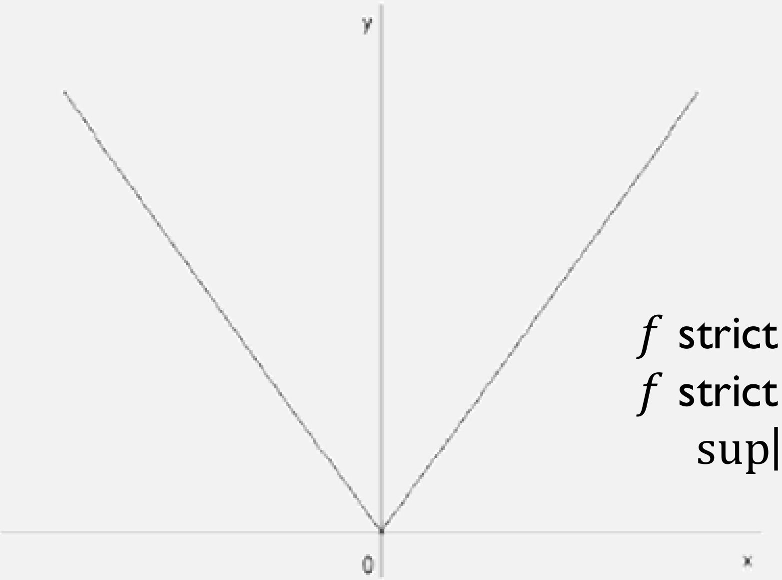
$$f(x) = x \text{ and } f(x) = -x$$

$$f(x) = |x|$$

$$f : x \in \mathbb{R} \to |x| \in [0, +\infty)$$

$$f(x) = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$





f strictly decreasing in $(-\infty, 0]$
f strictly increasing in $[0, +\infty)$
$\sup|x| = +\infty; \ \inf|x| = 0$

for a general function f:

$$|f(x)| = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ -f(x) \text{ if } f(x) < 0 \end{cases}$$

$$\frac{1}{x^2+|x-2|} = \begin{cases} \dfrac{1}{x^2+(x-2)} & \text{se} \quad (x-2) \geq 0 \\ \dfrac{1}{x^2+(-(x-2))} & \text{se} \quad (x-2) < 0 \end{cases}$$

- **ELEMENTARY FUNCTIONS**
  - **Power function**

Power function

Categories:

- Power function with **natural exponent**
  - Even natural exponent
  - Odd natural exponent

- Root function (**rational exponent**)
  - Rational exponent with even base
  - Rational exponent with odd base

- Power function with **real exponent**
  - $b > 0$
  - $b < 0$

Power function with natural exponent

$$f(x) = x^n, \qquad n \in \mathbb{N} \text{ fixed}$$

$$f : x \in \mathbb{R} \rightarrow x^n \in codomain\ depending\ on\ n$$

x is the **base** of the power function and varies in the domain
n is the **exponent** of the power function and is fixed

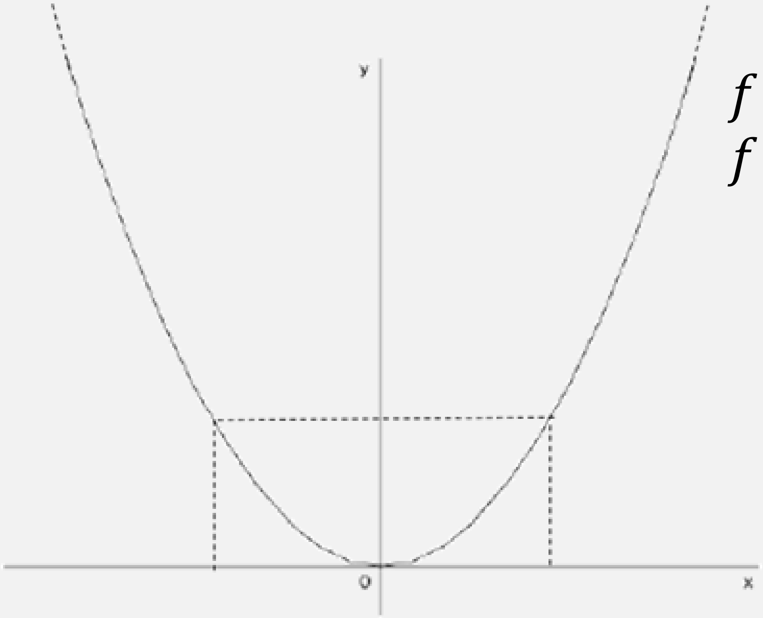
Rules:

Positive base: any exponent is admissible
Negative base: integer exponent or rational exponent with odd denominator

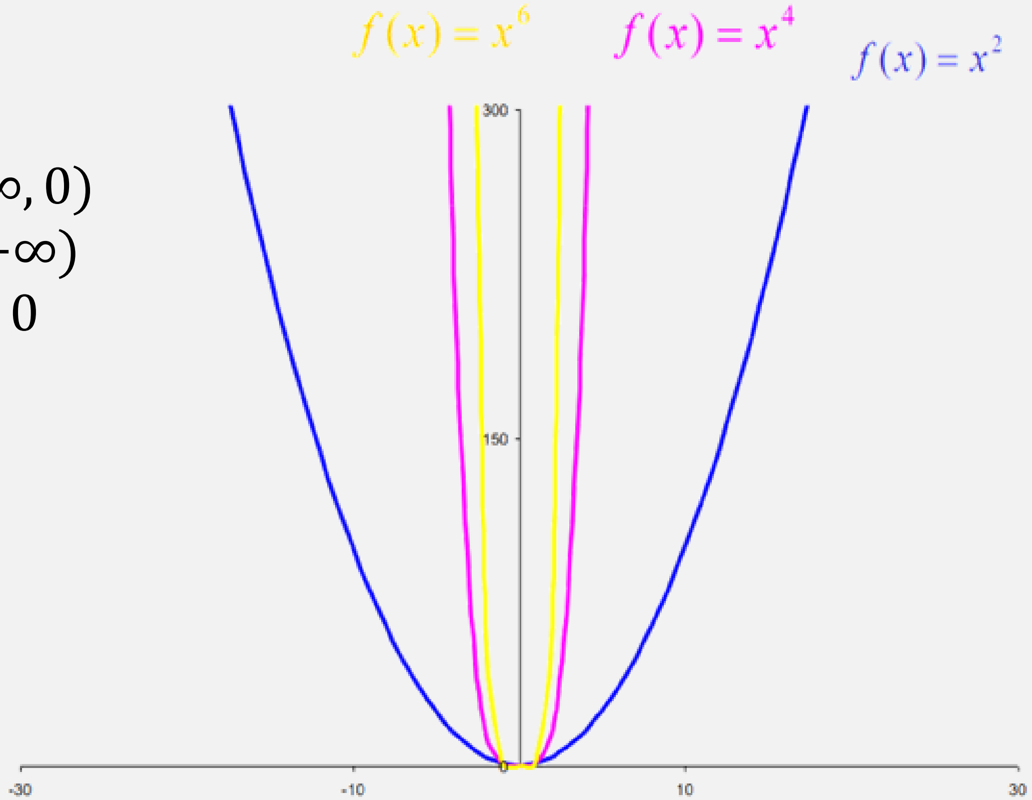
Power function with even natural exponent

$$f(x) = x^n, \qquad n \in \mathbb{N} \text{ fixed and even}$$

$$f : x \in \mathbb{R} \to x^n \in [0, +\infty)$$



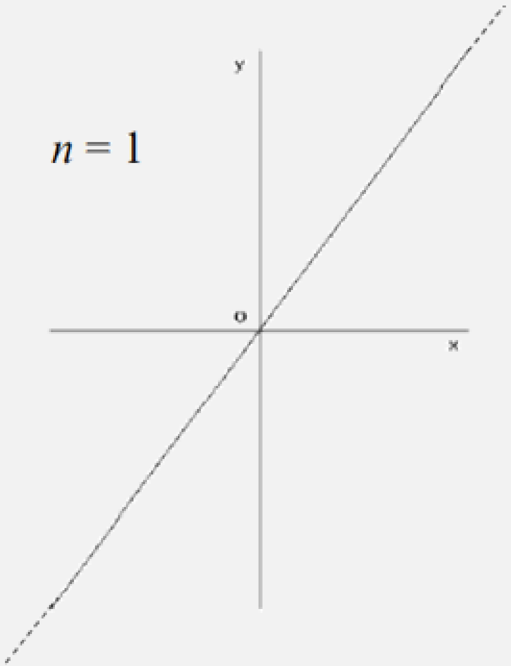
f strictly decreasing in $(-\infty, 0)$
f strictly increasing in $(0, +\infty)$
$\sup x^n = +\infty;\ \inf x^n = 0$

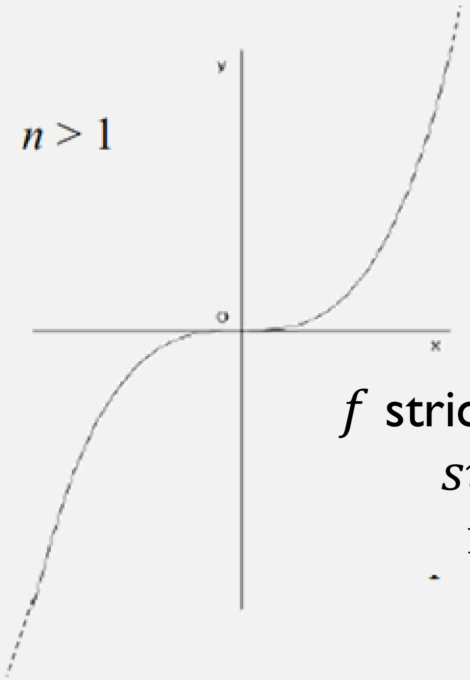


Power function with odd natural exponent

$$f(x) = x^n, \qquad n \in \mathbb{N} \text{ fixed and odd}$$

$$f : x \in \mathbb{R} \to x^n \in \mathbb{R}$$

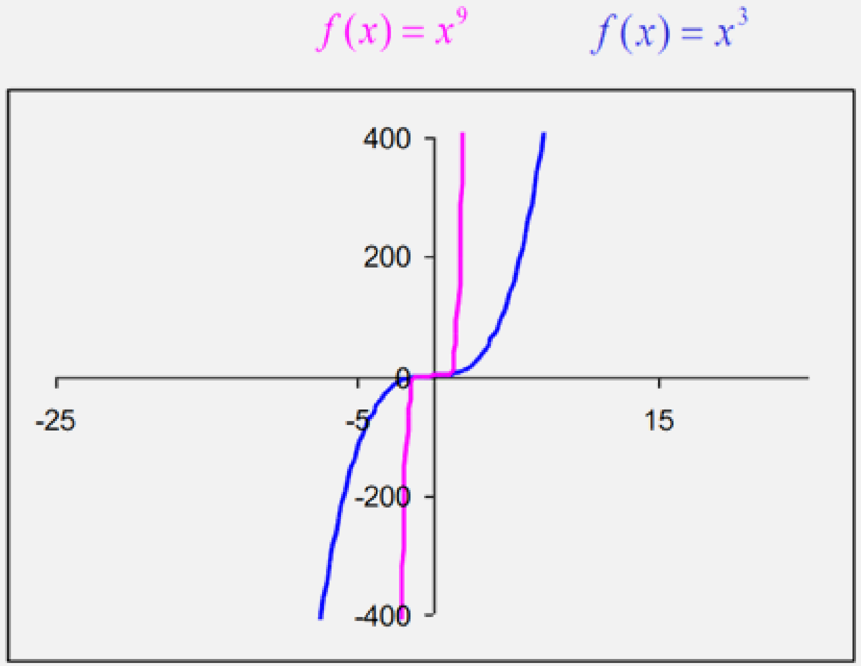




f strictly increasing in $\mathbb{R}$

$sup\, x^n = +\infty;$

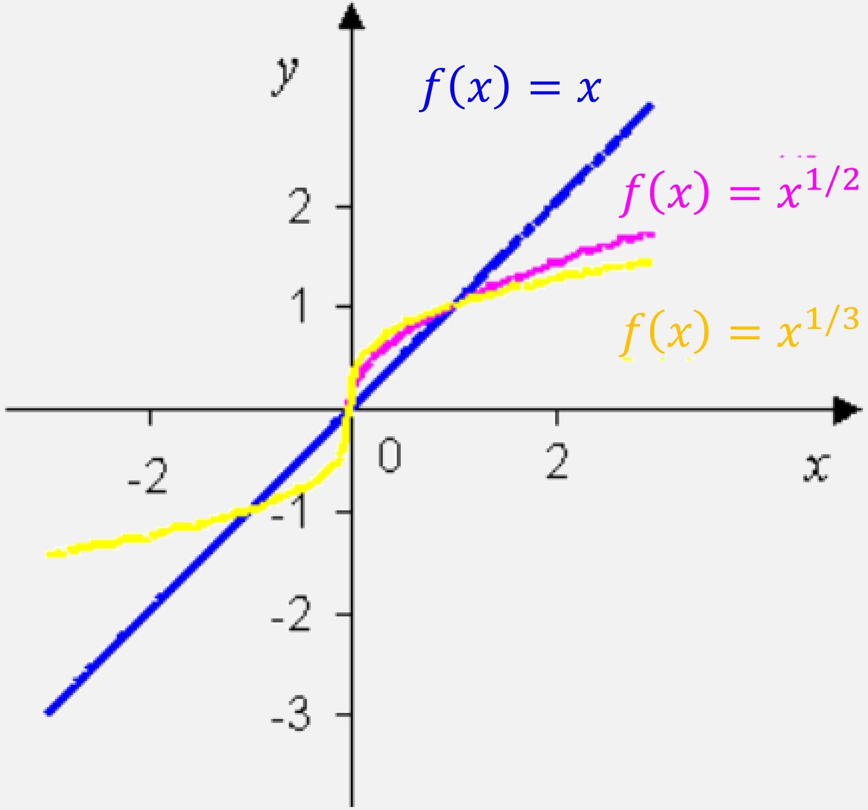
$\inf x^n = -\infty$



Power function with exponent $1/n$ (root function)

$$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}, \qquad n \in \mathbb{N} \text{ fixed}, n \geq 2$$

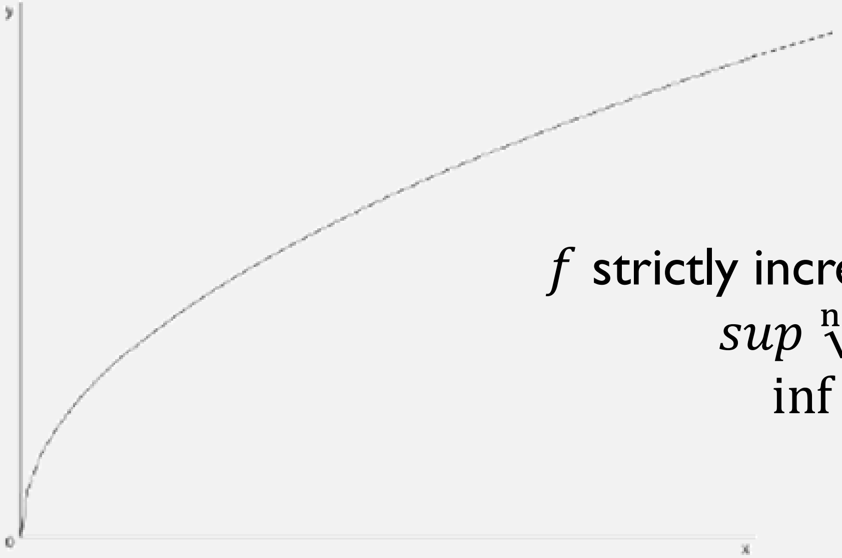
The domain and codomain depend on the value of n



Even root function

$$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}, \qquad n \in \mathbb{N} \text{ fixed}, n \geq 2 \text{ and even}$$

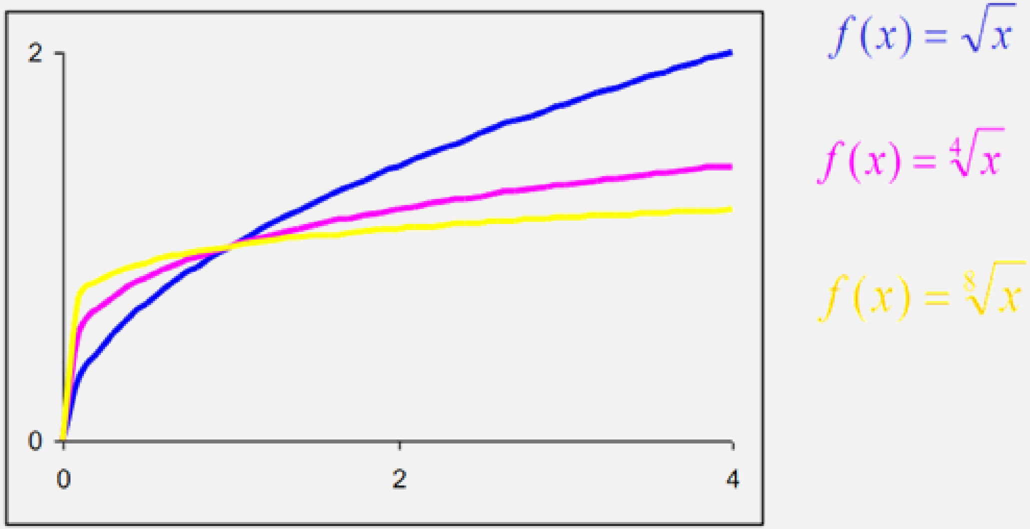
$$f : x \in [0, +\infty) \to \sqrt[n]{x} \in [0, +\infty)$$



f strictly increasing in $[0, +\infty)$

$sup \sqrt[n]{x} = +\infty;$

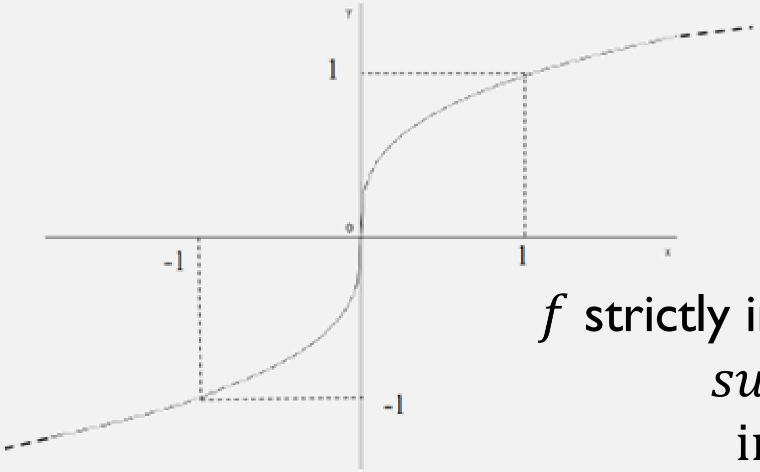
$\inf \sqrt[n]{x} = 0$



Odd root function

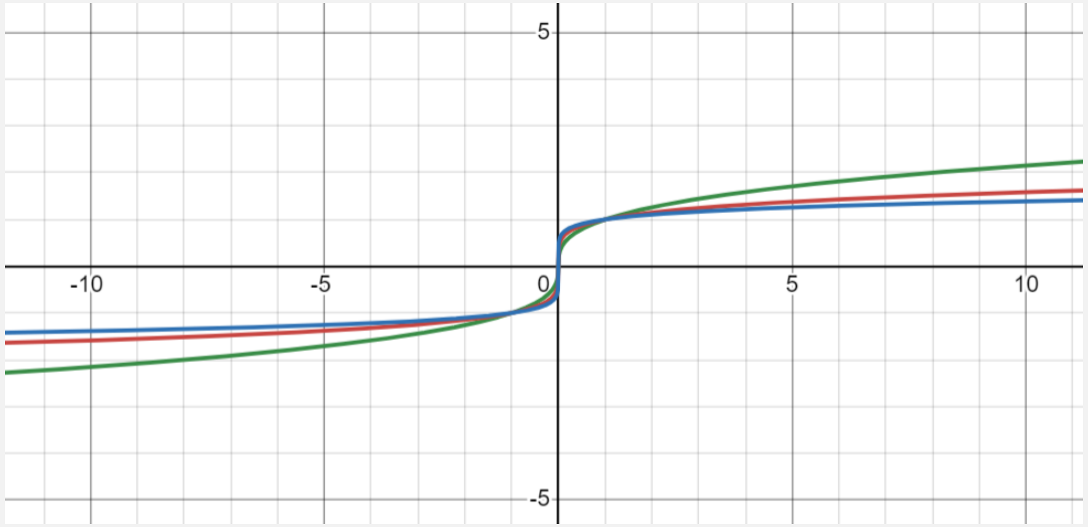
$$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}, \qquad n \in \mathbb{N} \text{ fixed}, n \geq 2 \text{ and odd}$$

$$f : x \in \mathbb{R} \rightarrow \sqrt[n]{x} \in \mathbb{R}$$



f strictly increasing in $\mathbb{R}$

$$sup \sqrt[n]{x} = +\infty;$$
$$\inf \sqrt[n]{x} = -\infty$$



$f(x) = \sqrt[3]{x}$

$f(x) = \sqrt[5]{x}$

$f(x) = \sqrt[7]{x}$

Power function with real exponent

$$f(x) = x^b, \qquad b \in \mathbb{R} \text{ fixed}$$

$$The\ domain\ and\ codomain\ depend\ on\ the\ value\ of\ b$$

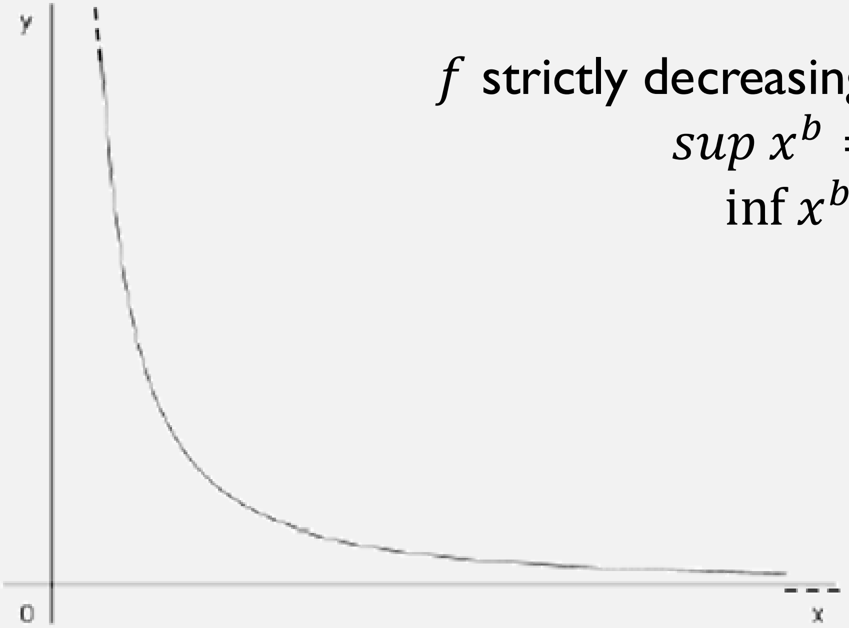
x is the base and varies in the domain

b is the real exponent and is fixed

Power function with real esponent $b < 0$

$$f(x) = x^b, \qquad b \in \mathbb{R} \text{ fixed}, b < 0$$

$$f : x \in (0, +\infty) \to x^b \in (0, +\infty)$$



f strictly decreasing in $(0, +\infty)$

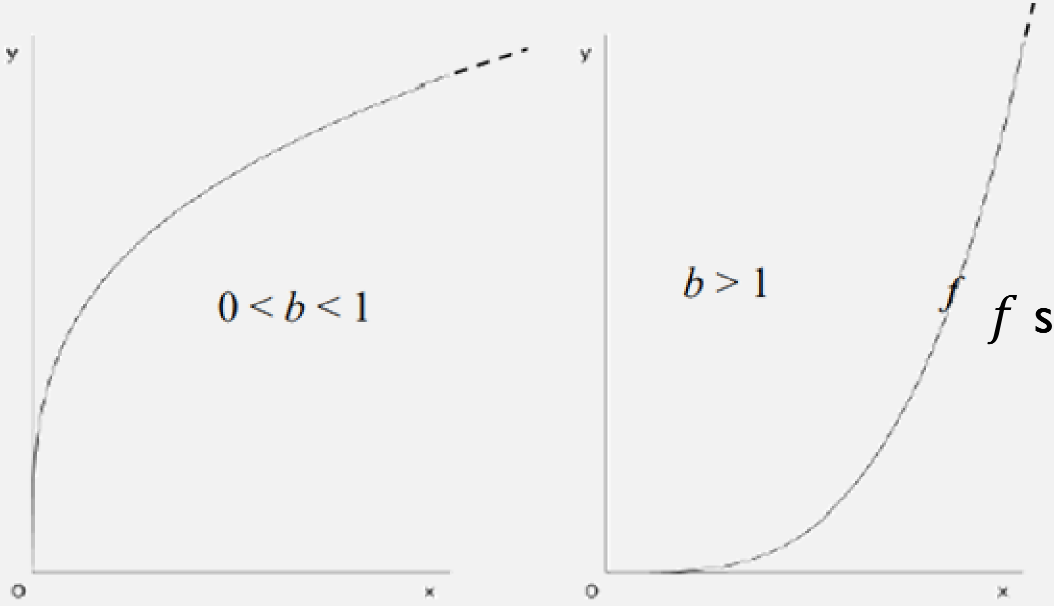
$sup\, x^b = +\infty;$

$\inf x^b = 0$

Power function with real esponent $b > 0$

$$f(x) = x^b, \qquad b \in \mathbb{R} \text{ fixed}, b > 0$$

$$f : x \in [0, +\infty) \to x^b \in [0, +\infty)$$



f strictly increasing in $(0, +\infty)$

$$sup\, x^b = +\infty;$$

$$\inf x^b = 0$$

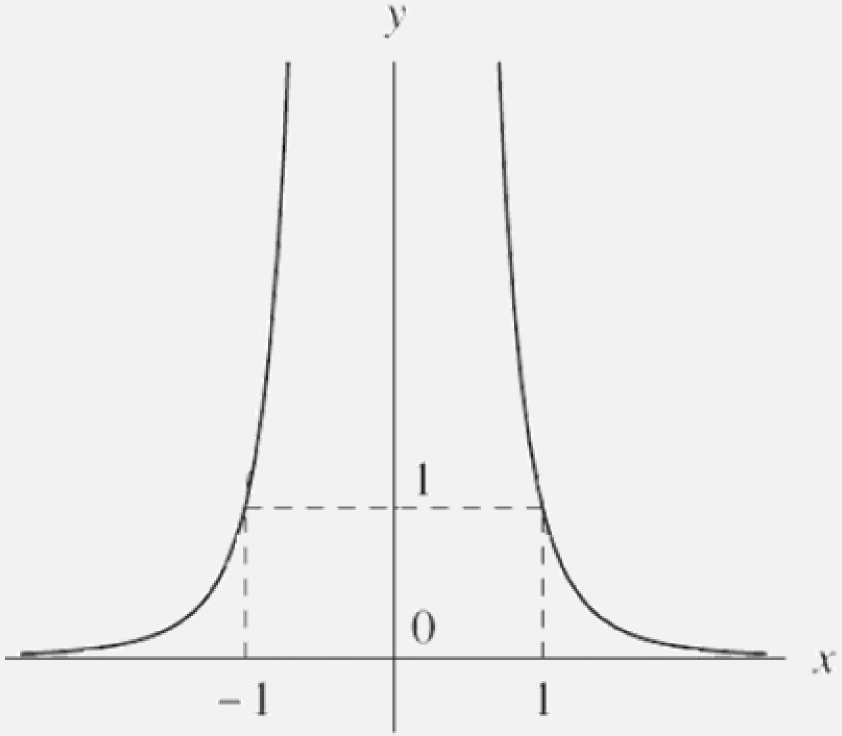
Power function with negative even integer exponent

By definition:

$$f(x) = x^{-n} = \left(\frac{1}{x}\right)^{n} = \frac{1}{x^{n}}$$

$$f(x) = x^{n}, n \in \mathbb{Z} \text{ fixed}, n \text{ even}, n < 0$$

$$f : x \in \mathbb{R} - \{0\} \rightarrow x^{n} \in (0, +\infty)$$



f strictly increasing in $(-\infty, 0)$

f strictly decreasing in $(0, +\infty)$

$$sup\ x^{\mathrm{n}} = +\infty;$$

$$\inf x^{\mathrm{n}} = 0$$

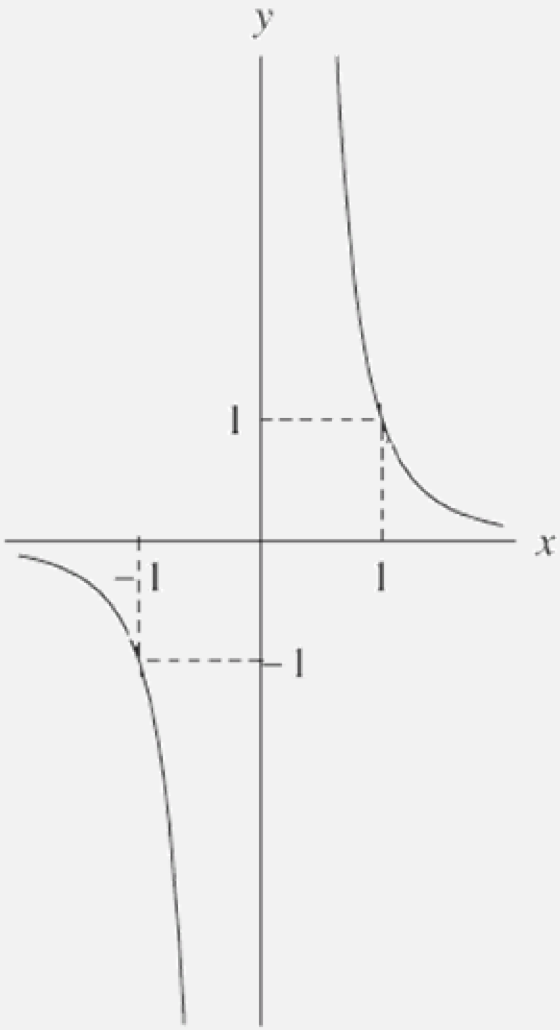
Power function with negative odd integer exponent

By definition:

$$f(x) = x^{-n} = \left(\frac{1}{x}\right)^{n} = \frac{1}{x^{n}}$$

$$f(x) = x^{n}, n \in \mathbb{Z}\ \text{fixed}, n\ \text{odd}, n < 0$$

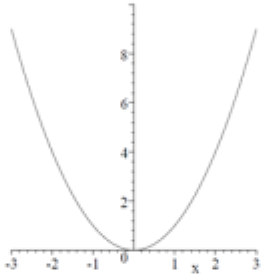
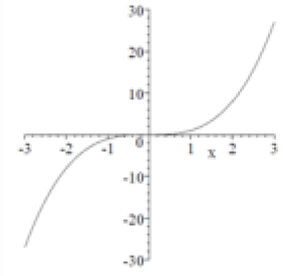
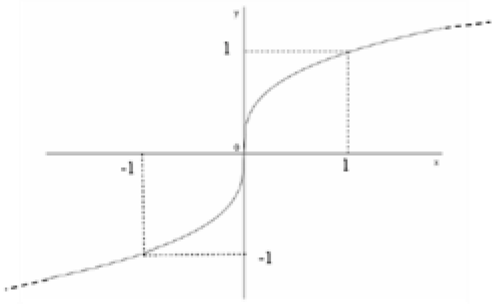
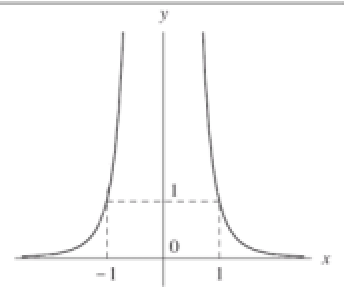
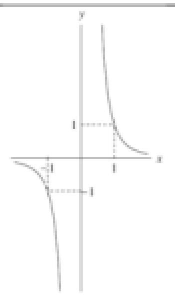
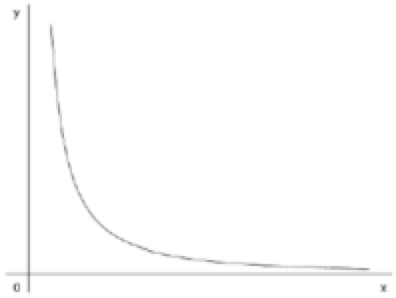
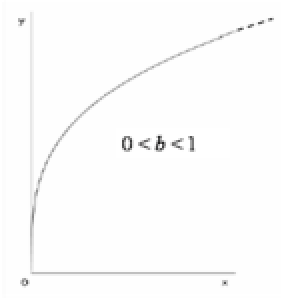
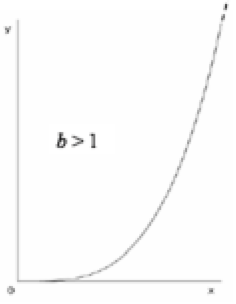
$$f : x \in \mathbb{R} - \{0\} \to x^{n} \in \mathbb{R} - \{0\}$$



f strictly decreasing in $\mathbb{R}$

$sup\ x^{\mathrm{n}} = +\infty;$

$\inf x^{\mathrm{n}} = -\infty$

	n even	n odd
x^n		
$X^{(1/n)}$		
x^n, $n<0$		
x^b, b reale	 b<0	 

- **ELEMENTARY FUNCTIONS**
  - **Polynomial function**

Polynomial/quadratic function

A **polynomial function** is defined as a (generally non-monotonic) function expressed by the general form:

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$
$$f : D \to \mathbb{R}$$

$n \in \mathbb{N}$ is the **degree** of the polynomial

$a_n, \ldots, a_1, a_0 \in \mathbb{R}$ are the coefficients

The leading coefficient a_n is $\neq 0$

If the degree of the polynomial defining f is equal to 2:

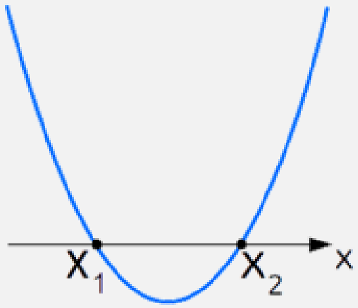
$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

- if $a > 0$, the parabola opens upwards, wihile if $a < 0$, lthe parabola opens downwards
- the abscissa of the vertex is $x = -\frac{b}{2a}$
- the vertex is $V = (-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$ and represents the symmetry axis of the parabola
- if $a > 0$ the ordinate of the vertex represents the minimum value of the function $f(x) = ax^2 + bx + c$, while if $a < 0$ it represents the maximum value
- the point $(0, c)$ is the intersection with the y-axis.
- The solutions of the equation $ax^2 + bx + c = 0$ give the intersections with the x-axis.

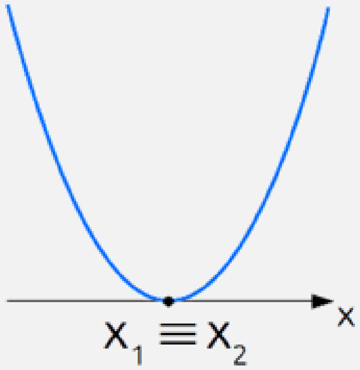
If the degree of the polynomial defining f is equal to 2**:**

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

- the **discriminant** is $\Delta = b^2 - 4ac$
- if $\Delta > 0$, the parabola intersects the x-axis at two distinct points x_1 and x_2. If the parabola opens upwards, then the graph will be above the x-axis for external values, meaning per $x < x_1$ and $x > x_2$
- if $\Delta = 0$, the parabola is tangent to the x-axis at one point. If the parabola opens upwards, the graph will always be above the x-axis, except for the contact point. If the parabola opens downwards, then the graph will always be below the x-axis
- if $\Delta < 0$, the parabola does not intersect the x-axis and, if it opens upwards, the graph will always be above the x-axis (viceversa if it opens downwards)

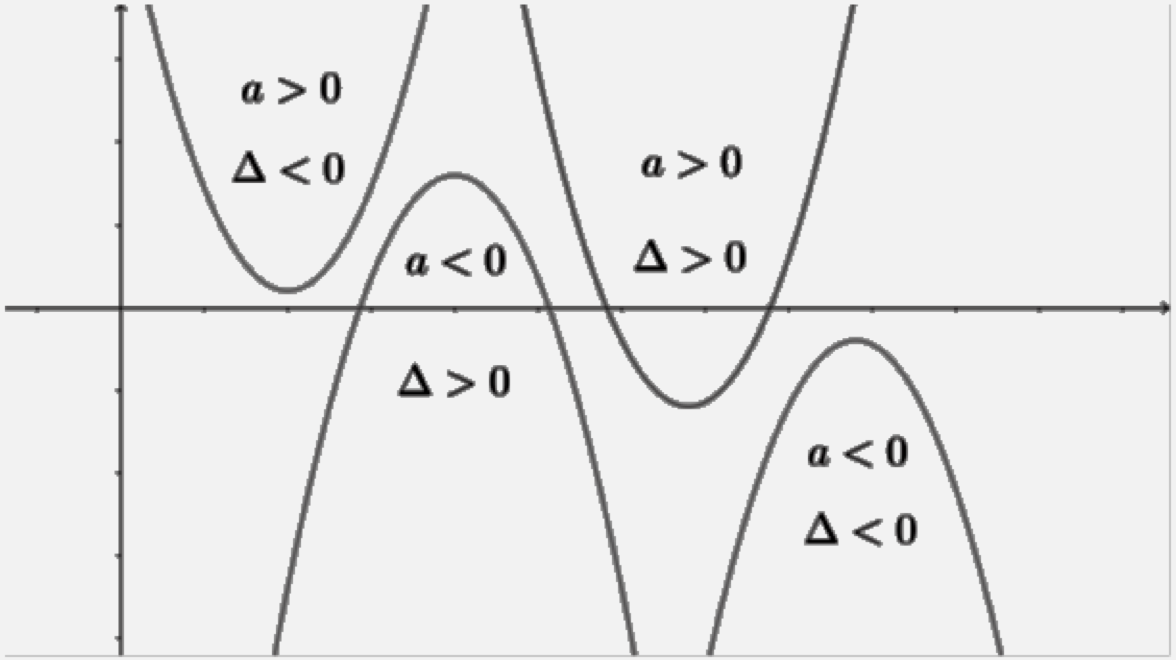


$\Delta > 0$



$\Delta = 0$

$\Delta < 0$



- **ELEMENTARY FUNCTIONS**
  - **Exponential function**

Exponential function

$$f(x) = a^x, a \in \mathbb{R} \text{ fixed}, a > 0, a \neq 1$$

$$f : x \in \mathbb{R} \to a^x \in (0, +\infty)$$

a is the **base** of the exponential function and is fixed
x is the **exponent** and varies in the domain

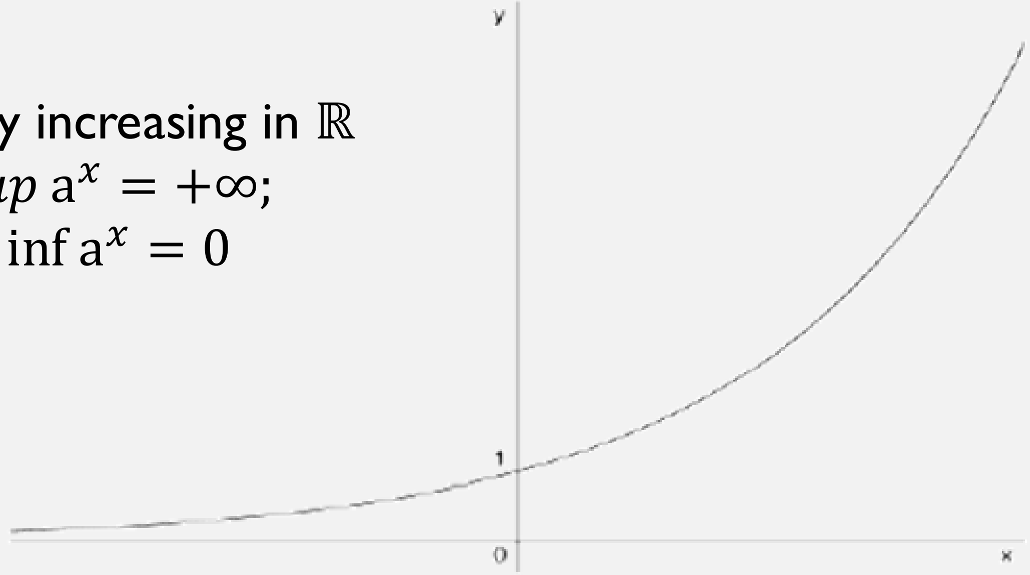
The exponential function is **always positive**

Exponential function with base $a > 1$

$$f(x) = a^x, a \in \mathbb{R}, a > 0, a \neq 1$$

case $a > 1$

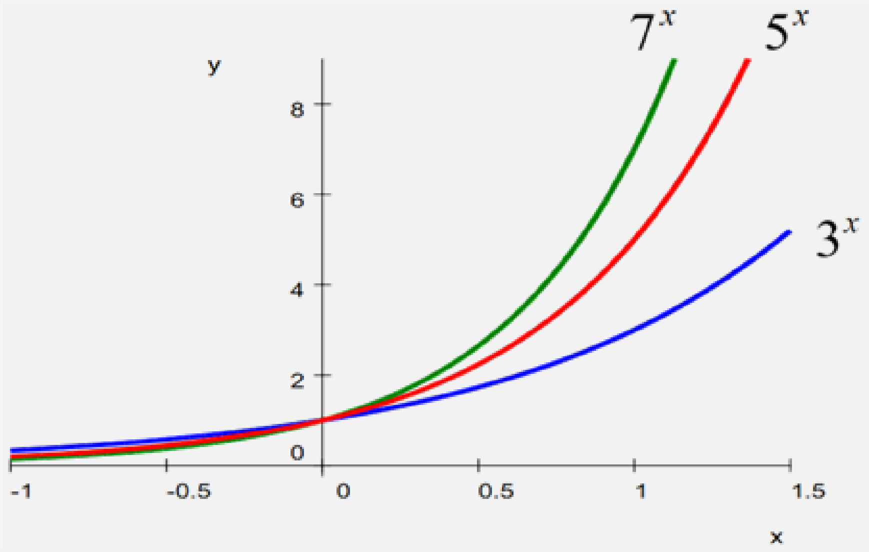
$$f : x \in \mathbb{R} \to a^x \in (0, +\infty)$$



f strictly increasing in $\mathbb{R}$

$sup\, \mathrm{a}^x = +\infty;$

$\inf \mathrm{a}^x = 0$

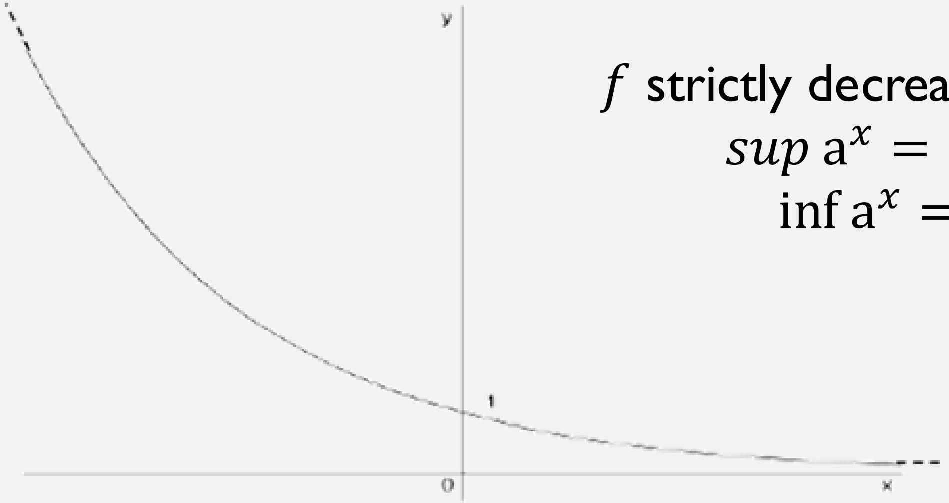


Exponential function with base $0 < a < 1$

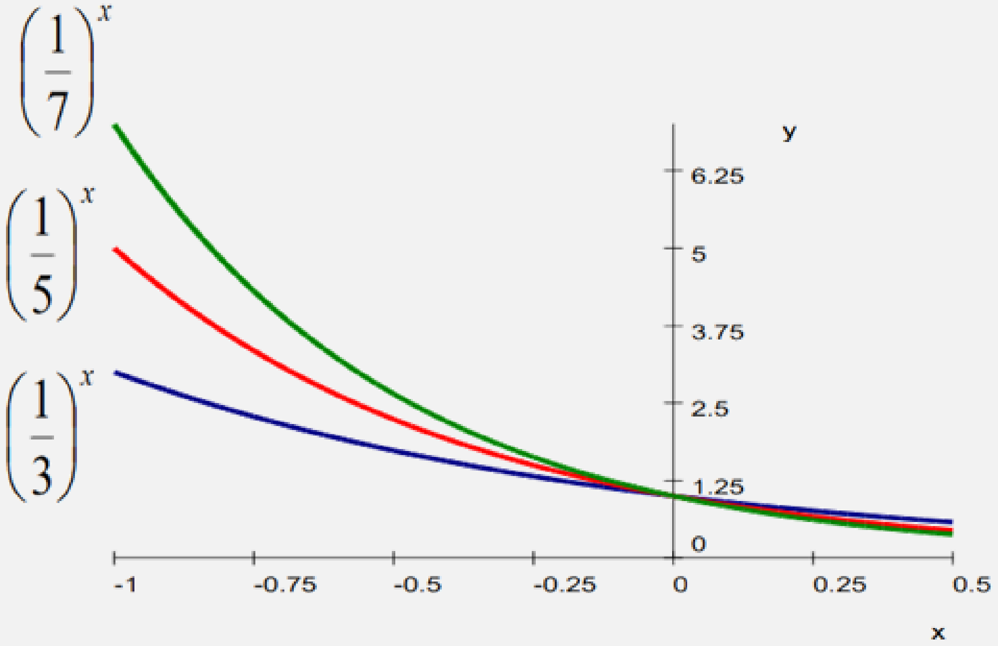
$$f(x) = a^x, a \in \mathbb{R}, a > 0, a \neq 1$$

case $0 < a < 1$

$$f : x \in \mathbb{R} \to a^x \in (0, +\infty)$$



f strictly decreasing in $\mathbb{R}$
$sup\ \mathrm{a}^x = +\infty$;
$\inf \mathrm{a}^x = 0$



Exponential function with base e

A particularly important exponential function has as base the irrational number

$$e = 2.7182$$

called **Napier's number**

The exponential function $f(x) = e^x$ is strictly increasing in $\mathbb{R}$ since $e > 1$

- **ELEMENTARY FUNCTIONS**
  - **Logarithmic function**

Logarithmic function

$$f(x) = \log_a x\,,\, a \in \mathbb{R} \text{ fixed}, a > 0, a \neq 1$$

$$f : x \in (0, +\infty) \to \log_a x \in \mathbb{R}$$

a is the **base** of the logarithm
x is the **argument** and varies within the domain

$$a^x = b \;\Leftrightarrow x = \log_a b$$

The logarithm is the inverse function of the exponential:

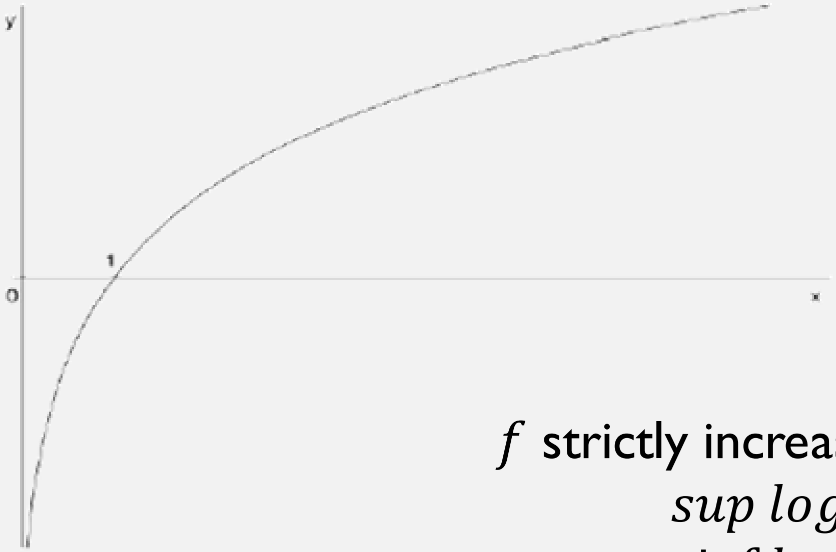
$$x = a^{\log_a x} \qquad x = \log_a(a^x)$$

Logarithmic function with $a > 1$

$$f(x) = \log_a x\,, a \in \mathbb{R}, a > 0, a \neq 1$$

case $a > 1$

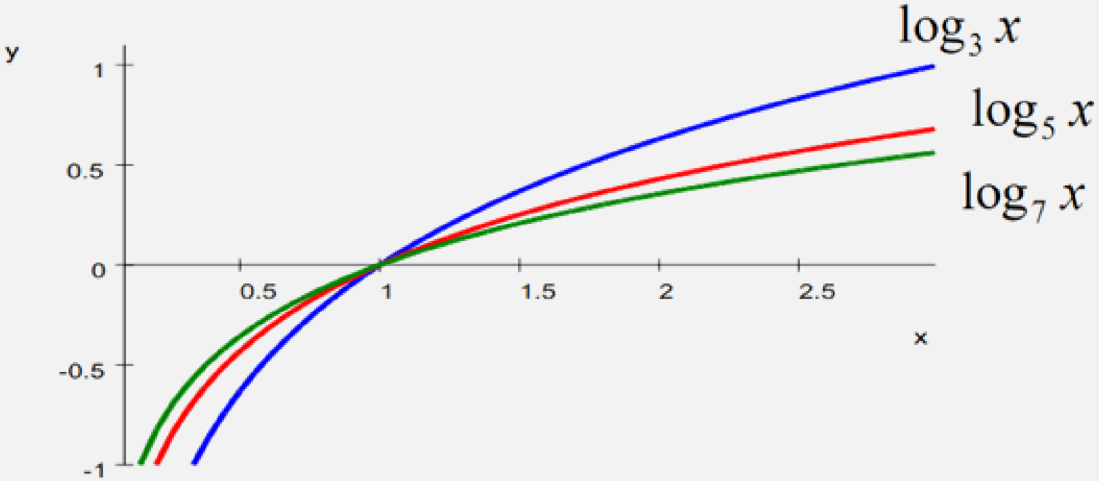
$$f : x \in (0, +\infty) \rightarrow \log_a x \in \mathbb{R}$$



f strictly increasing in $(0, +\infty)$

$$sup\ log_a \mathrm{x} = +\infty;$$

$$\inf log_a \mathrm{x} = -\infty$$

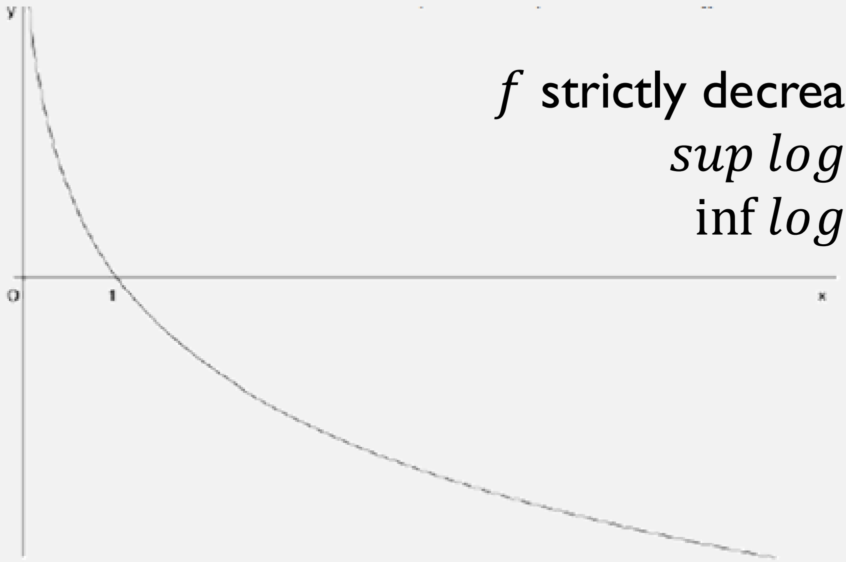


Logarithmic function with $0 < a < 1$

$$f(x) = \log_a x\,, a \in \mathbb{R}, a > 0, a \neq 1$$

case $0 < a < 1$

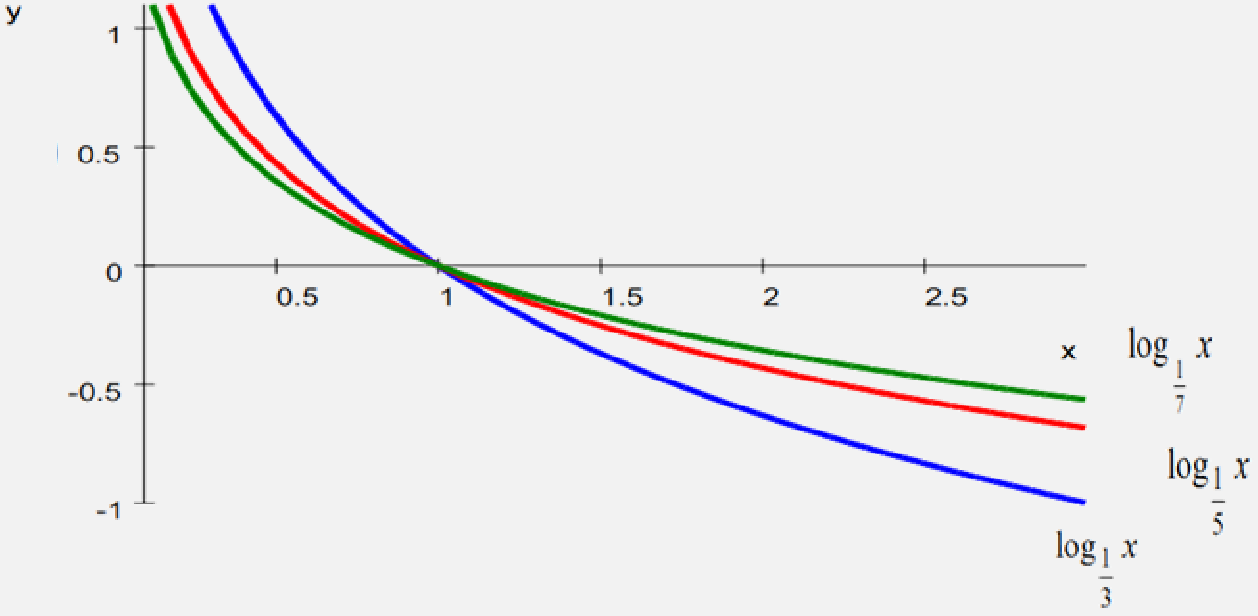
$$f : x \in (0, +\infty) \rightarrow \log_a x \in \mathbb{R}$$



f strictly decreasing in $(0, +\infty)$

$sup\, log_a \mathrm{x} = +\infty;$

$\inf log_a \mathrm{x} = -\infty$



Logarithmic function with base e – Natural Logarithm

The logarithmic function with base e is called the natural logarithm:

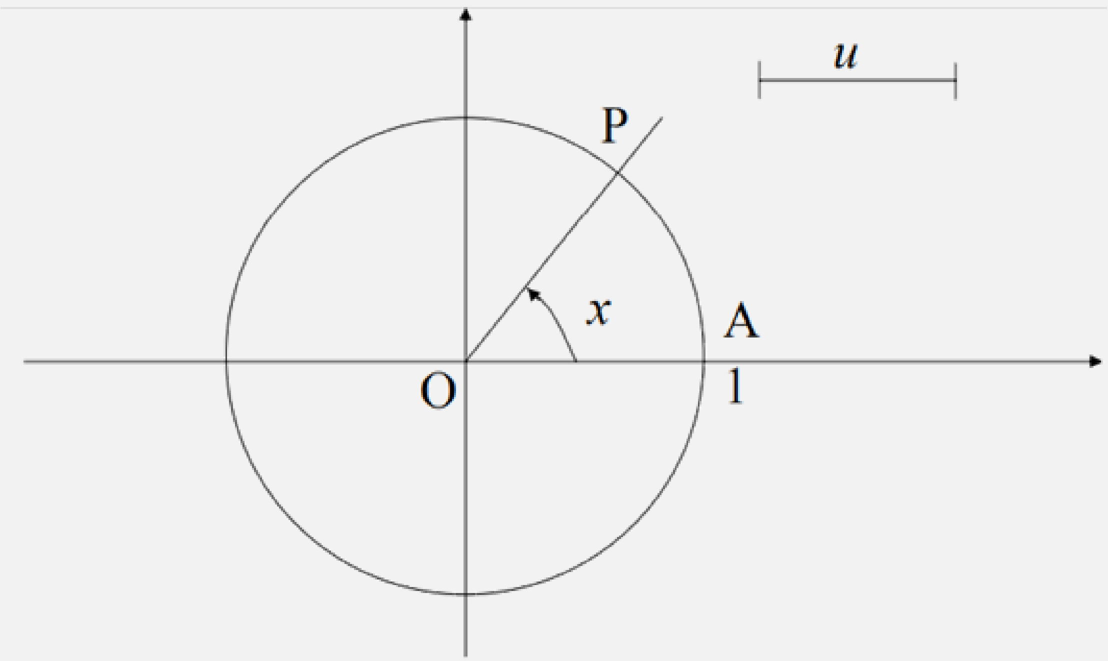
$$f(x) = \log_e x = \log x = \ln x$$

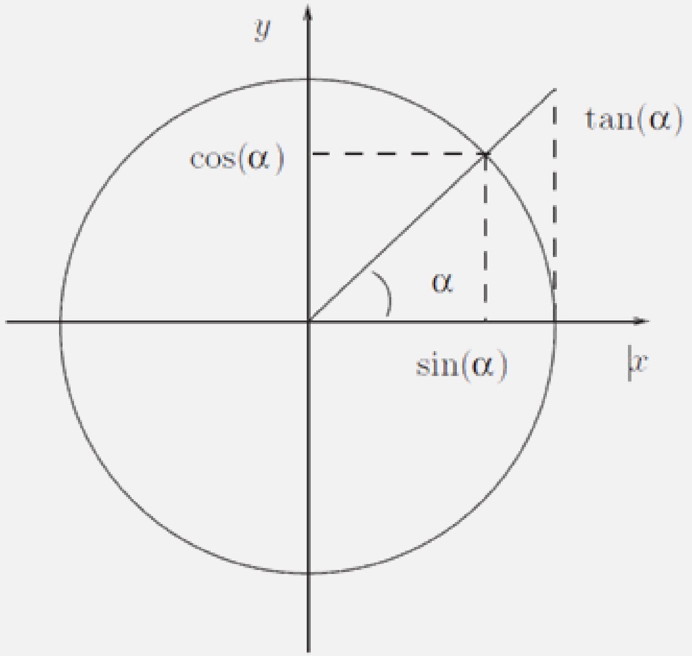
La funzione esponenziale $f(x) = \log_e x$ è una funzione strettamente crescente poiché $e > 1$

- **ELEMENTARY FUNCTIONS**
  - **Trigonometric functions**

Unit circle

Let P be a point moving on the unit circle, starting from point A, origin of arcs, and let x be the angle subtended by P.

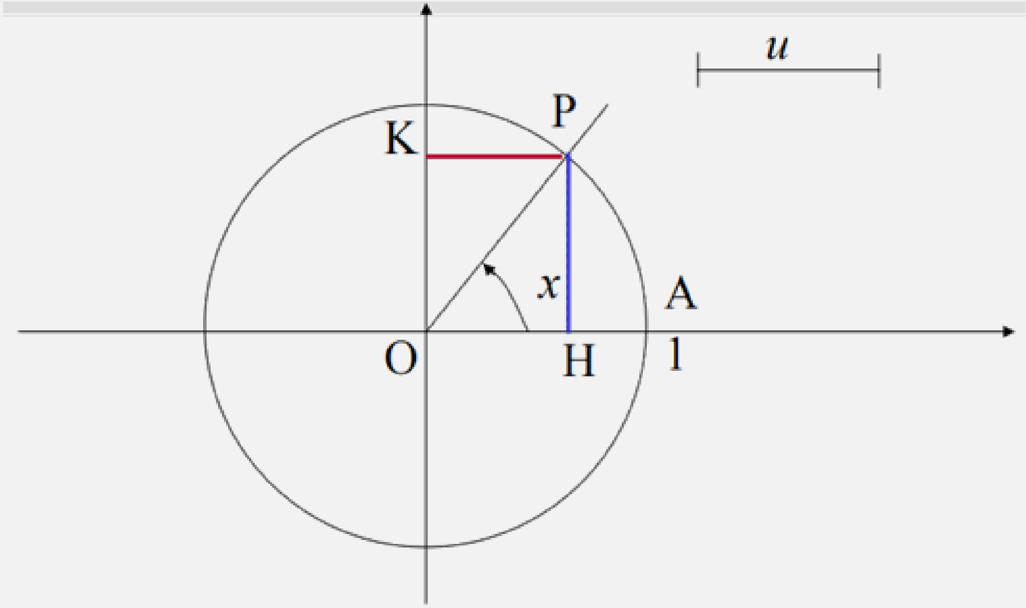


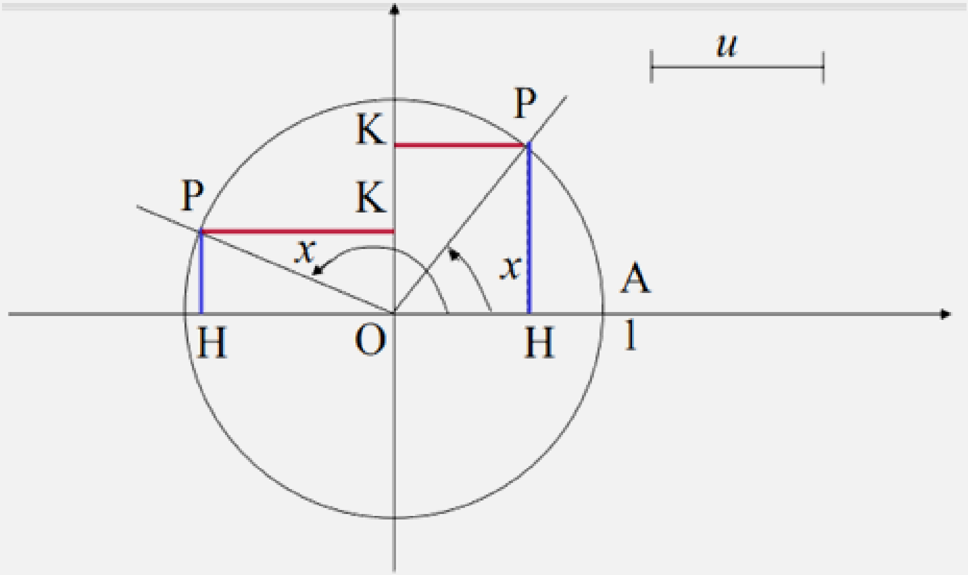


Fundamental identity of trigonometry: $\cos^2 x + \sin^2 x = 1$

Sine and cosine

$\sin x$ is defined as the ordinate of point P (meaning the segment PH).
$\cos x$ is defined as the abscissa of point P (meaning the segment PK).





As the position of P varies, the angle x, and hence $\sin x$ and $\cos x$, vary.

Sine function

$$f(x) = \sin x$$

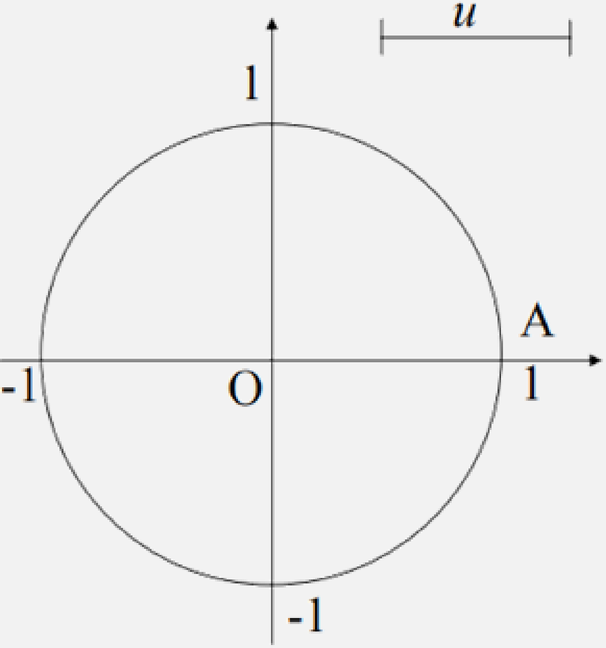
$$f : x \in \mathbb{R} \rightarrow \sin x \in [-1, +1]$$

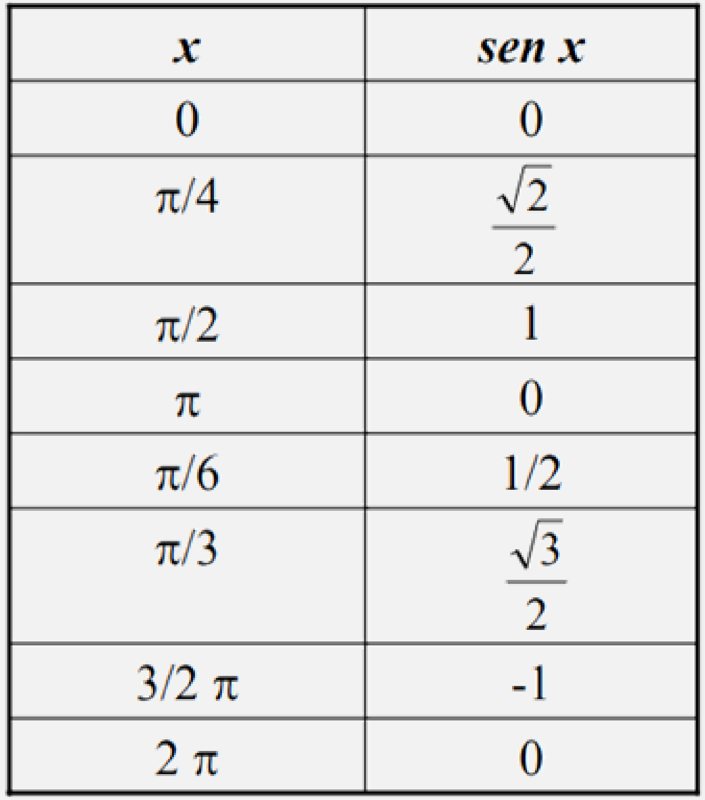
The **domain** of the sine function is the whole set $\mathbb{R}$, since a point P moving on the unit circle can traverse it infinitely many times, performing infinitely many revolutions (at each revolution, the angle increases by an amount equal to 2π).

The **codomain** of the sine function is the interval $[-1, +1]$, since the minimum value attained by the ordinate of point P is -1, while the maximum value is $+1$.

Sine function

For each fixed value of the arc x, the sine function assumes a corresponding numerical value.





x	sen x
0	0
$\pi/4$	$\frac{\sqrt{2}}{2}$
$\pi/2$	1
π	0
$\pi/6$	$1/2$
$\pi/3$	$\frac{\sqrt{3}}{2}$
$3/2\ \pi$	-1
$2\ \pi$	0

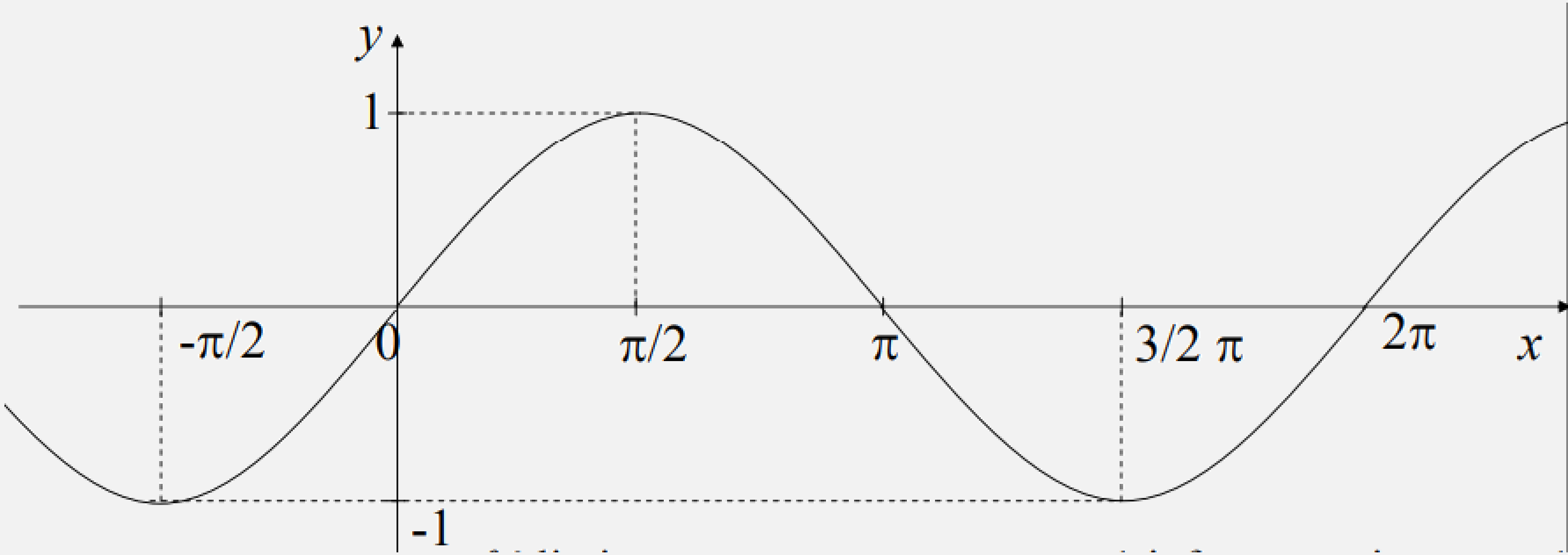
Periodicity
At each complete revolution (corresponding to an arc of legth 2π), the function $\sin x$ takes the same values

→ therefore, the sine function is <u>**periodic**</u> with period 2π:

$$\sin(x + 2\pi) = \sin x\,, \forall x \in \mathbb{R}$$

Notable points

- $(0{,}0) \rightarrow f(0) = 0$
- $\left(\frac{\pi}{2}, 1\right) \rightarrow f\left(\frac{\pi}{2}\right) = 1$
- $(\pi, 0) \rightarrow f(\pi) = 0$
- $\left(\frac{3\pi}{2}, -1\right) \rightarrow f\left(\frac{3\pi}{2}\right) = -1$
- $(2\pi, 0) \rightarrow f(2\pi) = 0$



f is bounded
f is strictly increasing in $[-\pi/2 + 2k\pi, \pi/2 + 2k\pi], k \in \mathbb{Z}$
f is strictly decreasing in $[\pi/2 + 2k\pi, 3/2\pi + 2k\pi], k \in \mathbb{Z}$

Cosine function

$$f(x) = \cos x$$

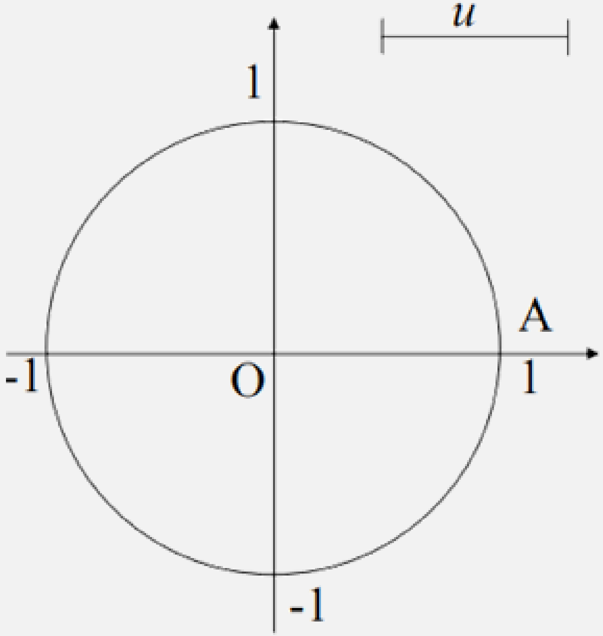
$$f : x \in \mathbb{R} \to \cos x \in [-1, +1]$$

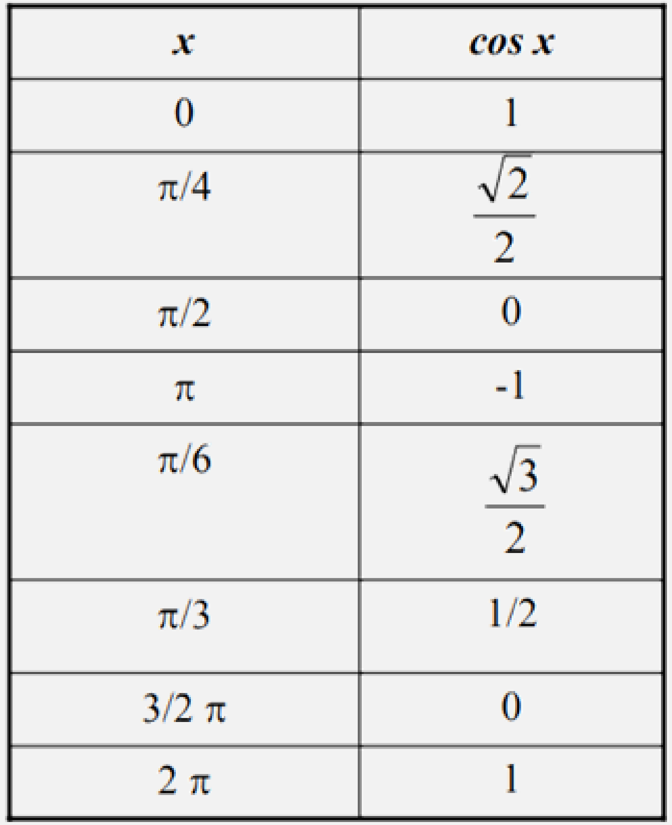
The **domain** of the cosine function is the whole set $\mathbb{R}$, since a point P moving on the unit circle can traverse it infinitely many times, performing infinitely many revolutions (at each revolution, the angle increases by an amount equal to 2π).

The **codomain** of the cosine function is the interval $[-1, +1]$, since the minimum value assumed by the ordinate of point P is -1, while the maximum value is $+1$.

Cosine function

For each fixed value of the arc x, the cosine function assumes a corresponding numerical value.



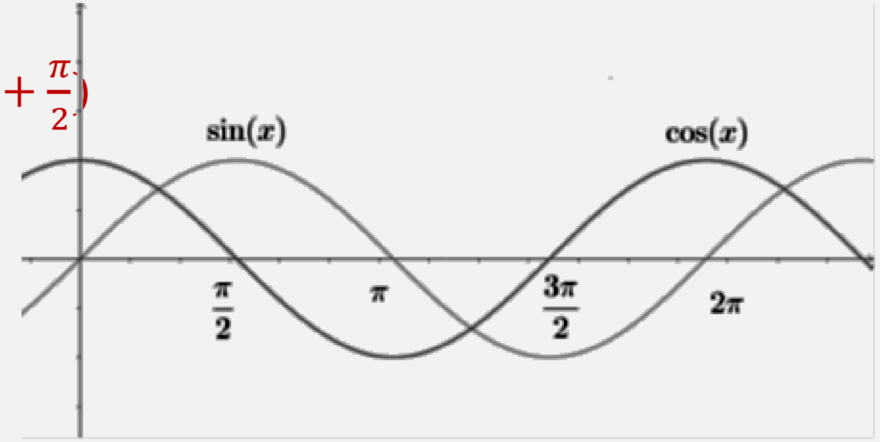


x	$\cos x$
0	1
$\pi/4$	$\frac{\sqrt{2}}{2}$
$\pi/2$	0
π	-1
$\pi/6$	$\frac{\sqrt{3}}{2}$
$\pi/3$	$1/2$
$3/2\,\pi$	0
$2\,\pi$	1

Periodicity
At each complete revolution (corresponding to an arc of length 2π), the function $\cos x$ takes the same values.
→ therefore, the cosine function is **periodic** of period 2π:

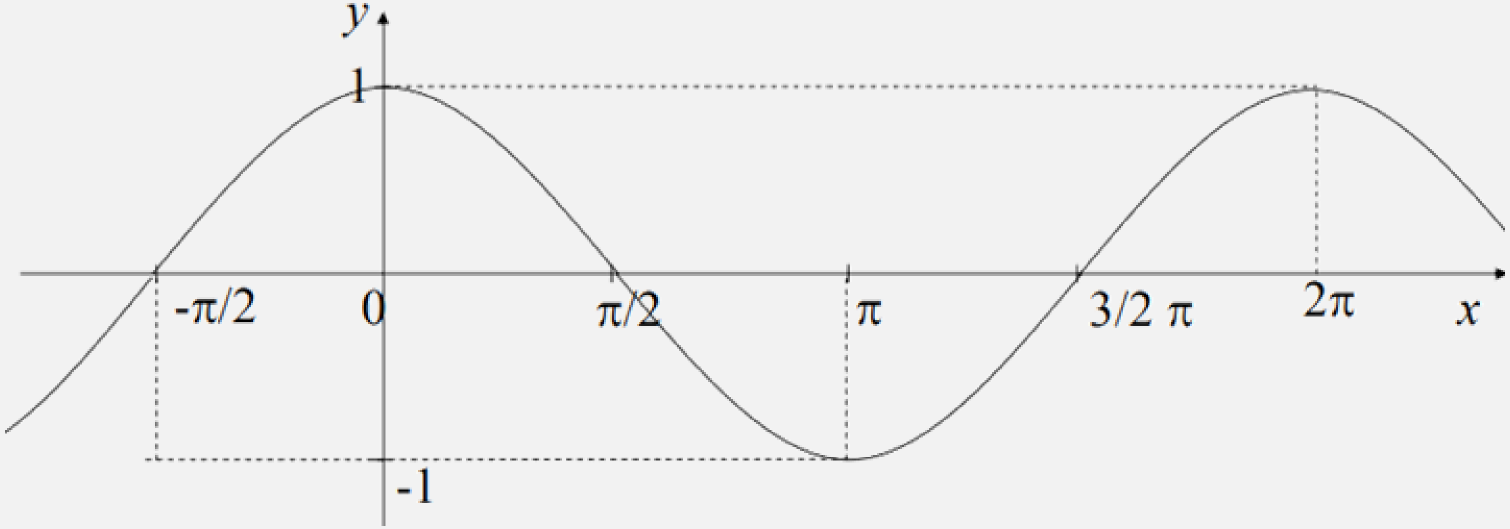
$$\cos(x + 2\pi) = \cos x\,, \forall x \in \mathbb{R}$$

The $\cos x$ function is equal to the $\sin x$ function shifted by $\frac{\pi}{2}$: $\cos x = \sin(x + \frac{\pi}{2})$



Notable points

- $(0,1) \rightarrow f(0) = 1$
- $\left(\frac{\pi}{2}, 0\right) \rightarrow f\left(\frac{\pi}{2}\right) = 0$
- $(\pi, -1) \rightarrow f(\pi) = -1$
- $\left(\frac{3\pi}{2}, 0\right) \rightarrow f\left(\frac{3\pi}{2}\right) = 0$
- $(2\pi, 1) \rightarrow f(2\pi) = 1$



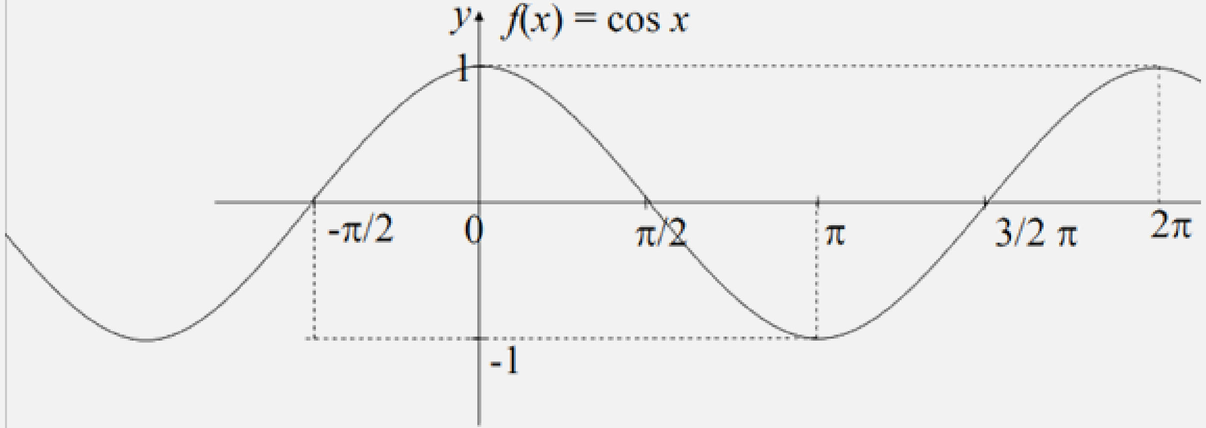
f is bounded
f is strictly increasing in $[\pi + 2k\pi, 2\pi + 2k\pi], k \in \mathbb{Z}$
f is strictly decreasing in $[2k\pi, \pi + 2k\pi], k \in \mathbb{Z}$

Tangent function

$$f(x) = \tan x = \operatorname{tg} x = \frac{\sin x}{\cos x}, \text{with} \cos x \neq 0$$

$$f : x \in \mathbb{R} - \left\{-\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots\right\}$$

The **domain** of the tangent function is the whole set $\mathbb{R}$ deprived of the values that make the cosine in the denominator equal to zero.



$$\cos x \neq 0 \ \text{se} \ x \neq \dots, -\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3}{2}\pi, \dots$$

$$\Downarrow$$

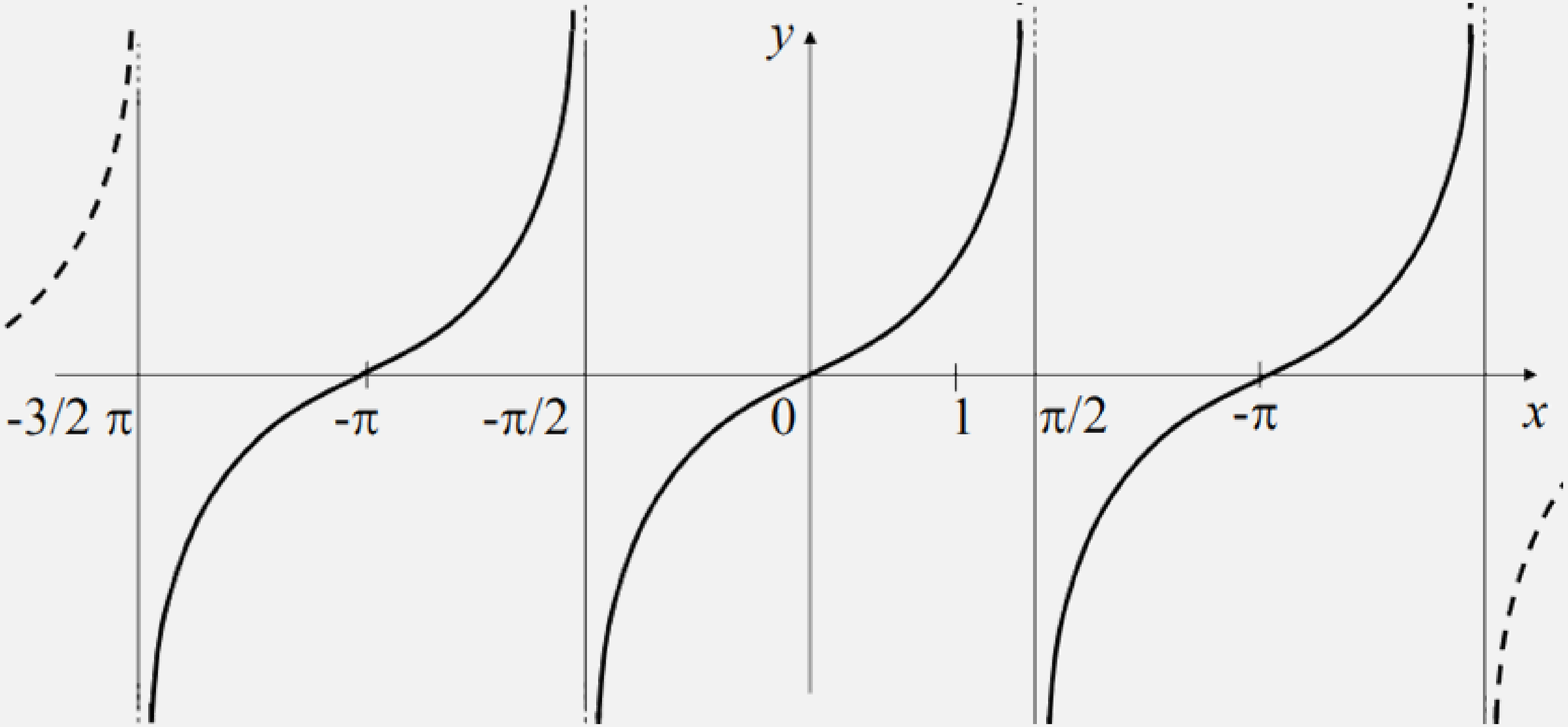
$$\cos x \neq 0 \ \text{se} \ x \neq \frac{\pi}{2} + k\pi, k \in Z$$

Tangent function

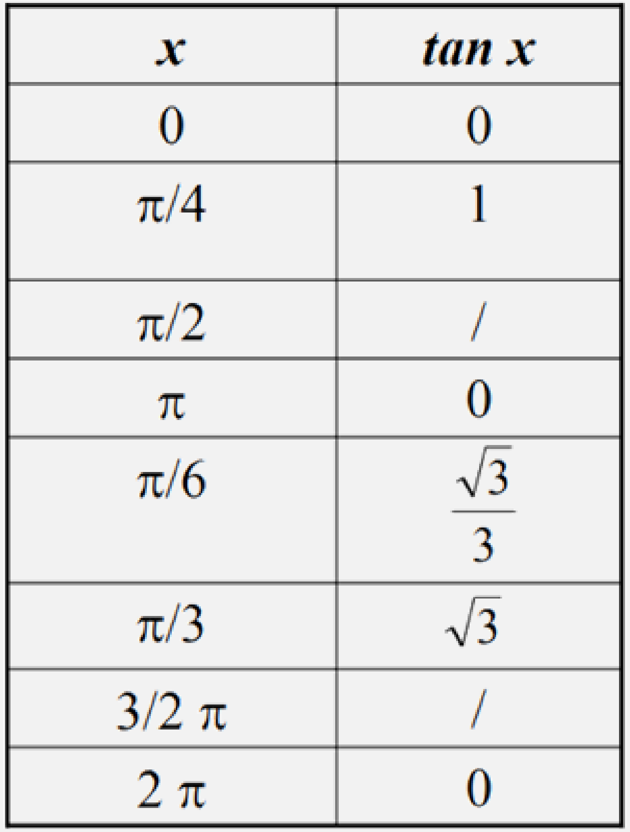
$$f(x) = \tan x = \mathrm{tg}\, x = \frac{\sin x}{\cos x}, \text{with} \cos x \neq 0$$

$$f: x \in \mathbb{R} - \left\{-\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots\right\} = f: x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\} \to \tan x \in \mathbb{R}$$

The **codomain** of the tangent function is $\mathbb{R}$



For each fixed value of the arc x, the tangent function assumes a corresponding numerical value.



x	$\tan x$
0	0
$\pi/4$	1
$\pi/2$	/
π	0
$\pi/6$	$\frac{\sqrt{3}}{3}$
$\pi/3$	$\sqrt{3}$
$3/2\ \pi$	/
$2\ \pi$	0

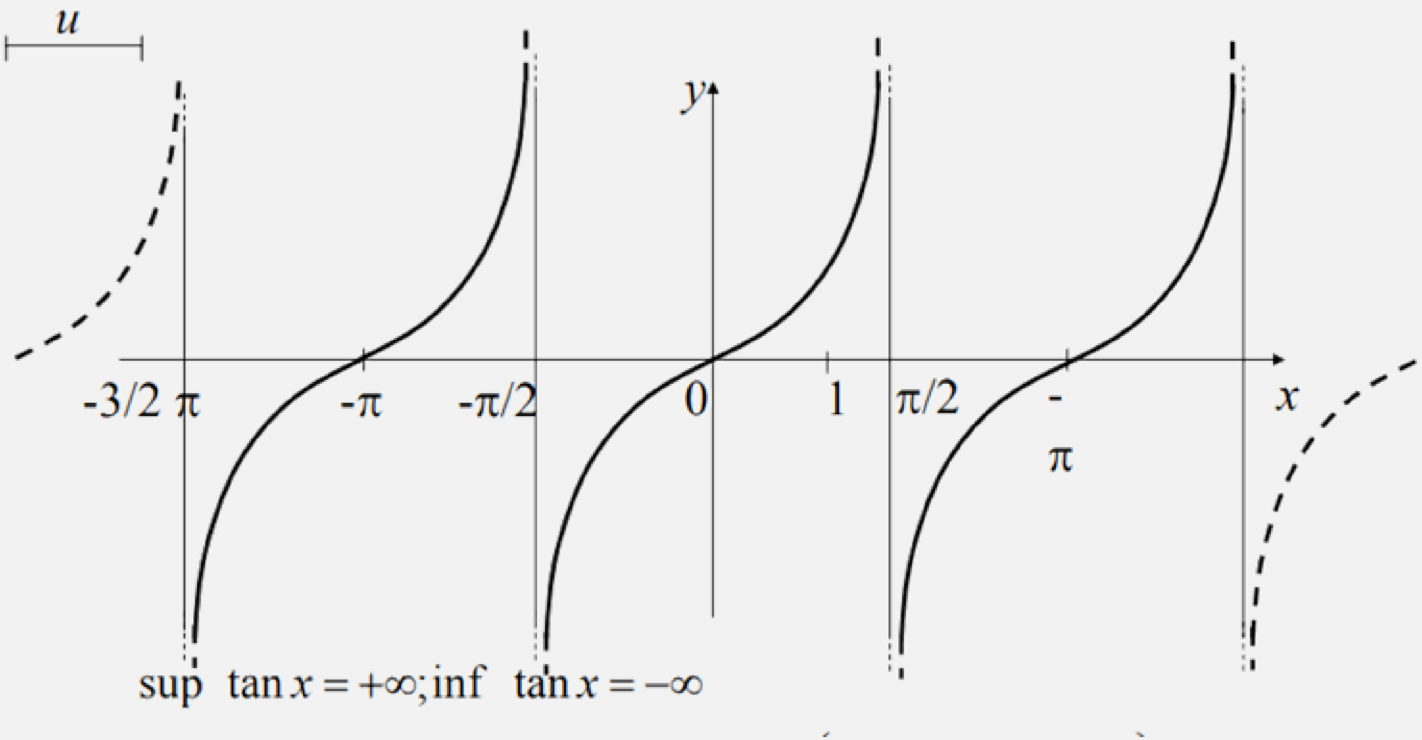
Periodicity
At each half revolution (corresponding to an arc of length π), the tangent function takes the same values.

→ therefore, the tangent function is **periodic** with period π:

$$\tan(x+\pi) = \tan x$$
$$\forall x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$$

Notable points

- $(0{,}0) \rightarrow f(0) = 0$
- $(\pi, 0) \rightarrow f(\pi) = 0$
- $(2\pi, 0) \rightarrow f(2\pi) = 0$



f is unbounded

f is strictly increasing in $(-\pi/2 + k\pi, \pi/2 + k\pi), \forall k \in \mathbb{Z}$

Since trigonometric functions are periodic, they are not injective.

However, they are strictly increasing or decreasing on suitable intervals: the restrictions to these intervals are **injective** and therefore **invertible**.

For each trigonometric function, a **fundamental interval** is chosen, that is, a set on which the restriction of the function is injective.

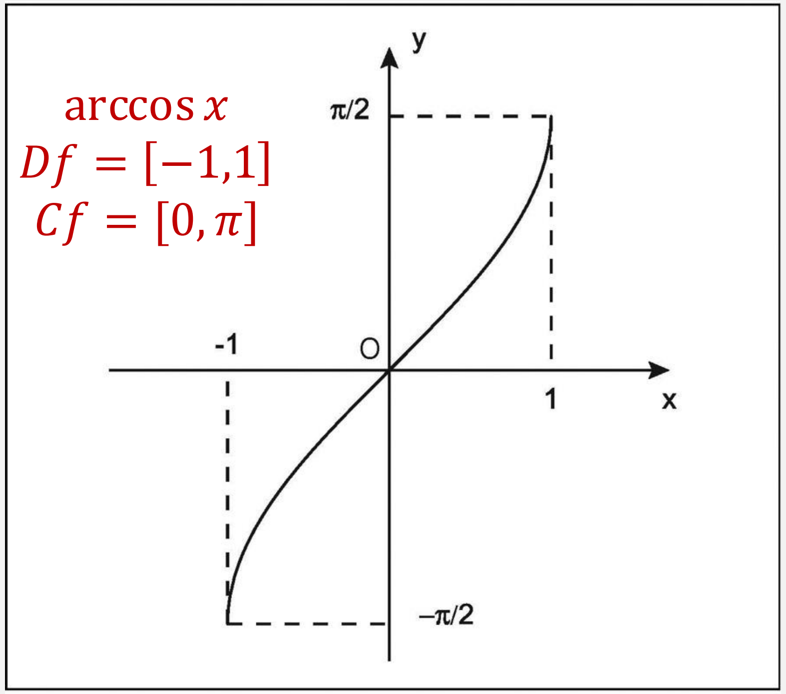
Within each fundamental interval, the inverse function can be defined.

Trigonometric function	*Fundamental interval*	*Inverse function*
$\sin x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\arcsin x$
$\cos x$	$[0, \pi]$	$\arccos x$
$\tan x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\arctan x$

The following identities hold (for all the trigonometric functions):

$\arcsin(\sin x) = x, \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

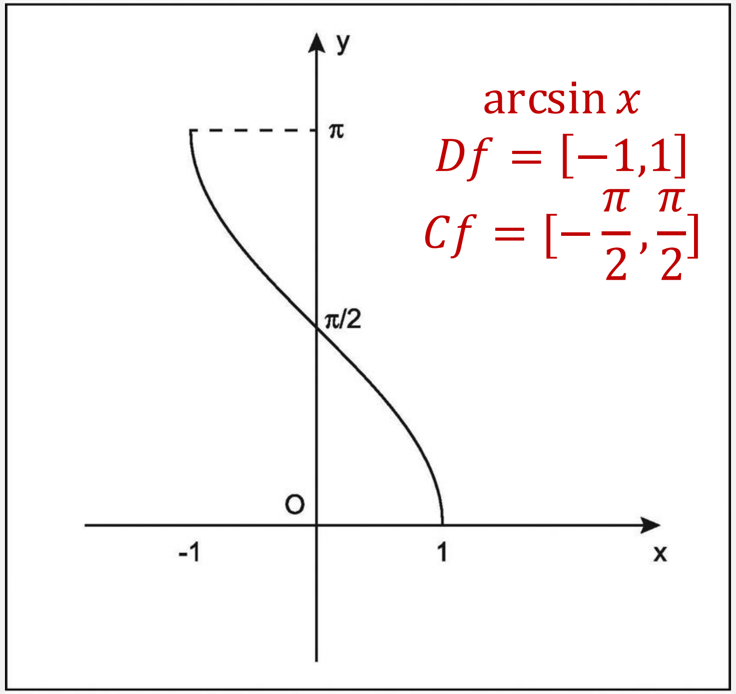
$\sin(\arcsin x) = x, \forall x \in [-1,1]$



$\arccos x$

$Df = [-1,1]$

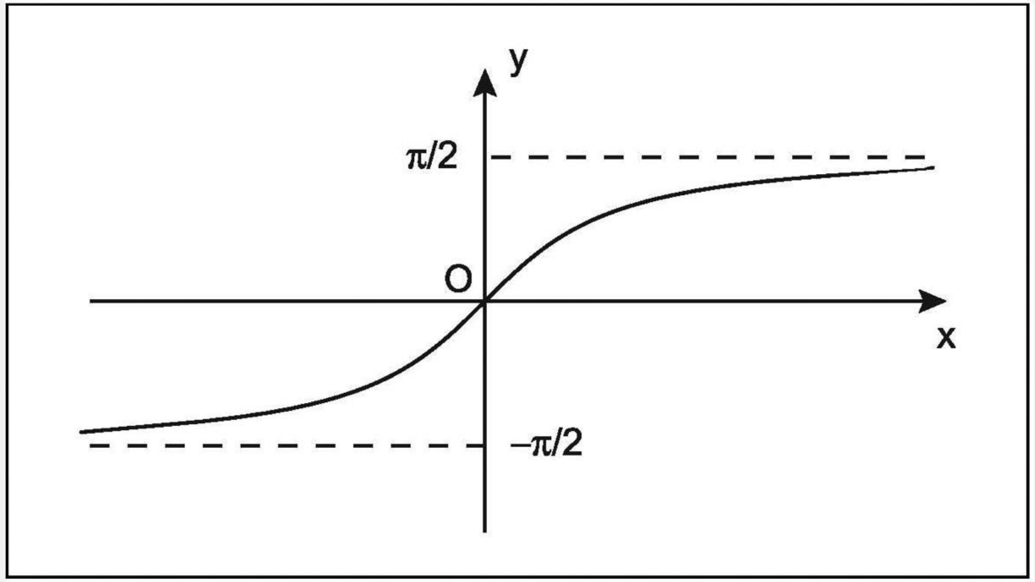
$Cf = [0, \pi]$



$\arcsin x$

$Df = [-1,1]$

$Cf = [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\arctan x$

$Df = \mathbb{R}$

$Cf = (-\frac{\pi}{2}, \frac{\pi}{2})$