- **LIMITS OF FUNCTIONS**

1. “Informal” definition of limit

$$\lim_{x \to x_0} f(x) = L$$

Let $f : D \to \mathbb{R}$ be a function defined for all x in a neighborhood of x_0, except possibly at $x = x_0$

We say that f tends to the limit L as x tends to x_0, if $f(x)$ takes values arbitrarily close to L whenever x is sufficiently close to x_0 (from both sides), excluding the point $x = x_0$.

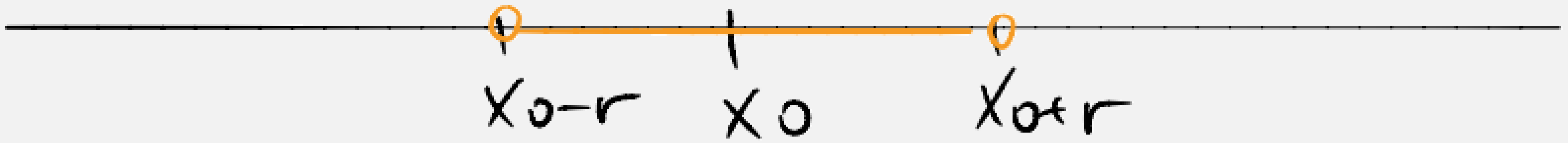
Recall: notion of neighborhood

Let $x_0 \in \mathbb{R}$ and $r > 0$.

We consider

$$I_r(x_0) = \{x \in \mathbb{R} : |x - x_0| < r\} = \{x \in \mathbb{R} : x_0 - r < x < x_0 + r\},$$

the open neighborhood of center x_0, that is, the set of real numbers whose distance from x_0 is less than $r > 0$.

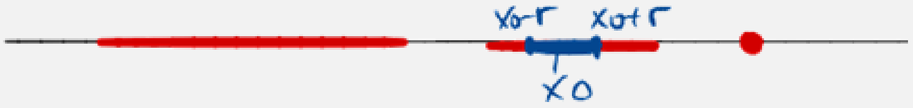


Recall: notion of neighborhood

Let $A \subset \mathbb{R}$ and $A \neq \emptyset$.

1. We say that $x_0 \in \mathbb{R}$ is an **interior point** of A if:

$$\exists r > 0 : I_r(x_0) \subset A$$



An interior point is a point for which there exists at least one neighborhood entirely contained in the set

Recall: notion of neighborhood

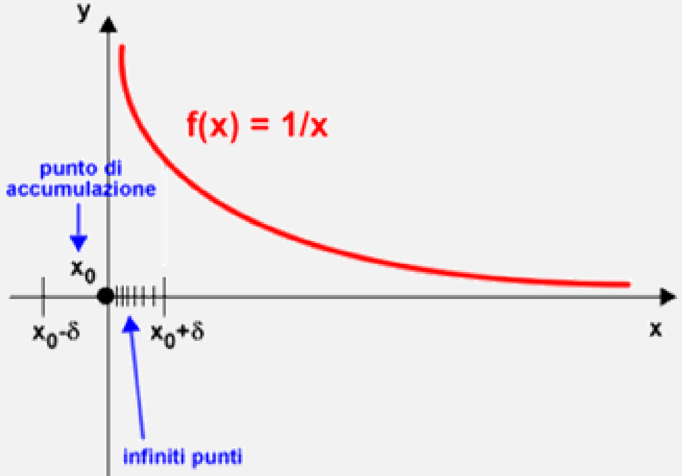
Let $A \subset \mathbb{R}$ and $A \neq \emptyset$.

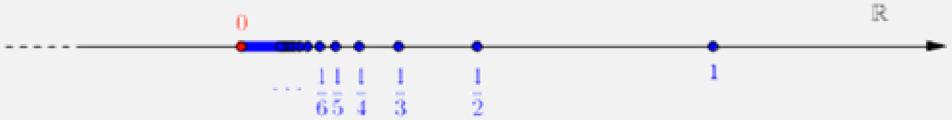
2. We say that $x_0 \in \mathbb{R}$ is an **accumulation point** of A if:

$$\forall r > 0 \ \rightarrow A - \{x_0\} \cap I_r(x_0) \neq \emptyset$$

That is, every neighborhood of x_0 contains at least one element of A different from x_0.

*An accumulation point is a point whose neighborhoods contain **infinitely many points** of A distinct from the point itself.*





L'insieme A è rappresentato in blu; 0 è un punto di accumulazione per A.

An accumulation point does not have to belong to A. For example:

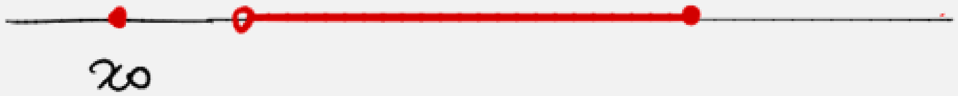
$$A = \{a \in \mathbb{R} \mid a = 1/n, n \in \mathbb{N}\}$$

this set admits as accumulation point $x = 0$: in fact, whatever neighborhood $B_\varepsilon(0) = (-\varepsilon, \varepsilon)$ is selected, we can find an n such that $1/n < \varepsilon$.
Moreover, $0 \notin A$, since no $1/n$ type of fraction can be equal to zero.

Recall: notion of neighborhood

Let $A \subset \mathbb{R}$ and $A \neq \emptyset$.

3. We say that $x_0 \in \mathbb{R}$ is an **isolated point** of A if:



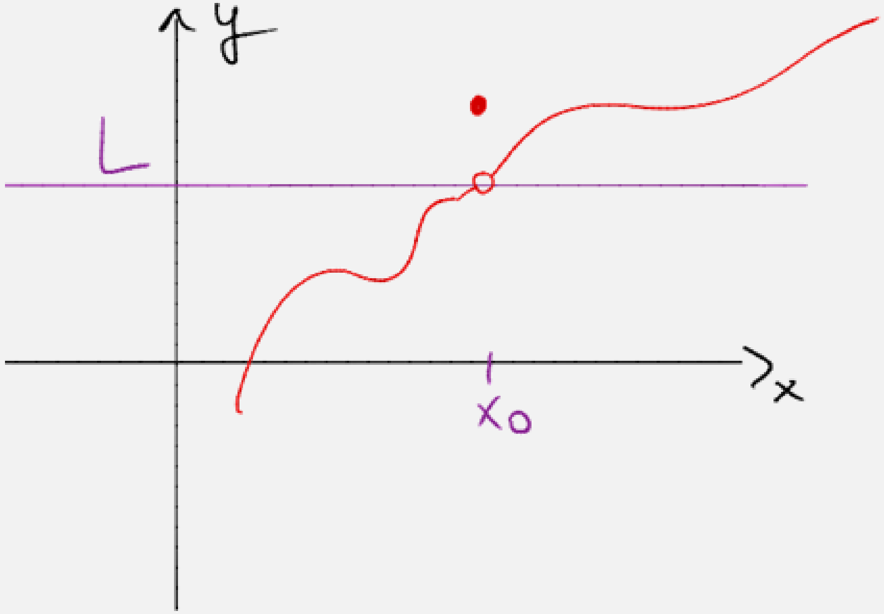
$$\exists r > 0 \ \rightarrow A \cap I_r(x_0) = \{x_0\}$$

An isolated point of a set A is a point belonging to the set for which there exists a neighborhood containing no other points of A.

4. We say that $x_0 \in \mathbb{R}$ is **adherent** to A if x_0 is either an accumulation point of A or an isolated point of A.

2. Definition of limit using neighborhoods

$$\lim_{x \to x_0} f(x) = L$$



with

- $L \in \mathbb{R}$

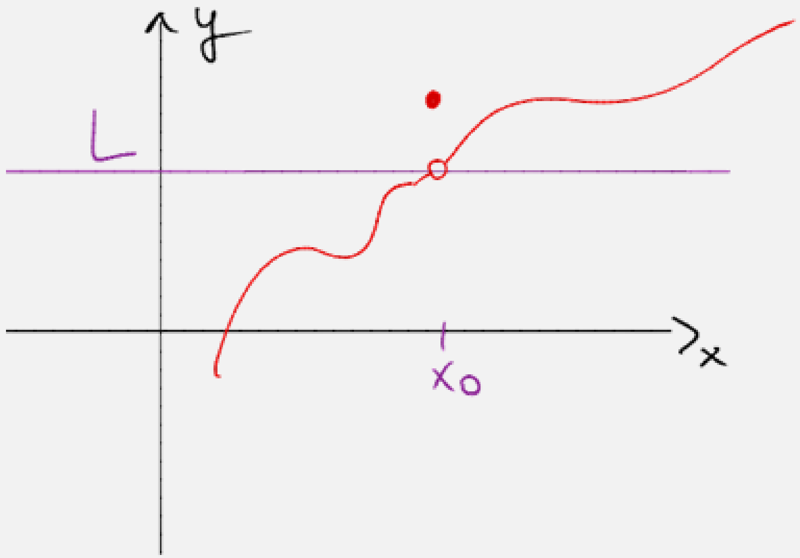
We say that the function f tends to the limit L as $x \to x_0$ if:

For every neighborhood $I(L, \varepsilon)$, with $\varepsilon > 0$, there exists a neighborhood $I(x_0, \delta)$, with $\delta > 0$, such that for all $x \in (I(x_0, \delta) - \{x_0\}) \cap D$ it follows that $f(x) \in I(L, \varepsilon)$

$$\forall I(L, \varepsilon), \varepsilon > 0 \; \exists \; I(x_0, \delta), \delta > 0 : \forall x \in (I(x_0, \delta) - \{x_0\}) \cap D \Longrightarrow f(x) \in I(L, \varepsilon)$$

3. "Rigorous" definition of limit

$$\lim_{x \to x_0} f(x) = L$$



Assume:

- $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$
- x_0 accumulation point of D
- $L \in \mathbb{R}$

We say that f tends to the limit L as $x \to x_0$ if:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : x \in (x_0 - \delta, x_0 + \delta) \Longrightarrow f(x) \in (L - \varepsilon, L + \varepsilon)$$

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; \forall x \in D, 0 < |x - x_0| \leq \delta, (x \neq x_0) \; \Longrightarrow |f(x) - L| \leq \varepsilon$$

The condition $x \neq x_0$ is imposed because the value of f at x_0 must not influence the limit

Cases of limits

General case:

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall I_L \; \exists \, I_{x0} : \; \forall x \in I_{x0} - \{x_0\} \Longrightarrow f(x) \in I_L$$

if and only if for every neighborhood of L there exists a neighborhood of x_0 such that, for all x belonging to the neighborhood of x_0 (possibly excluding x_0), $f(x)$ belongs to the neighborhood of L.

The cases may involve:

$$L = \begin{cases} \in \mathbb{R} \to I_L = (L - \varepsilon, L + \varepsilon) \\ +\infty \to I_L \quad = (n, +\infty) \\ -\infty \to I_L = (-\infty, -n) \end{cases}$$

$$x_0 = \begin{cases} \in \mathbb{R} \to I_L = (L - \varepsilon, L + \varepsilon) \\ +\infty \to I_L \quad = (n, +\infty) \\ -\infty \to I_L = (-\infty, -n) \end{cases}$$

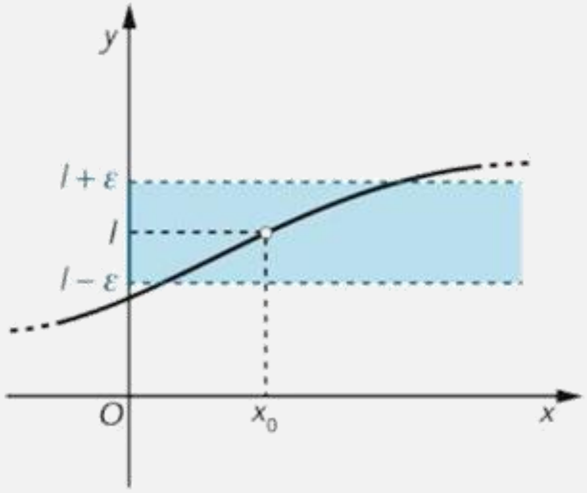
- **Finite limit at a finite point**:

  $L, x_0 \in \mathbb{R} \quad \lim_{x \to x_0} f(x) = L$

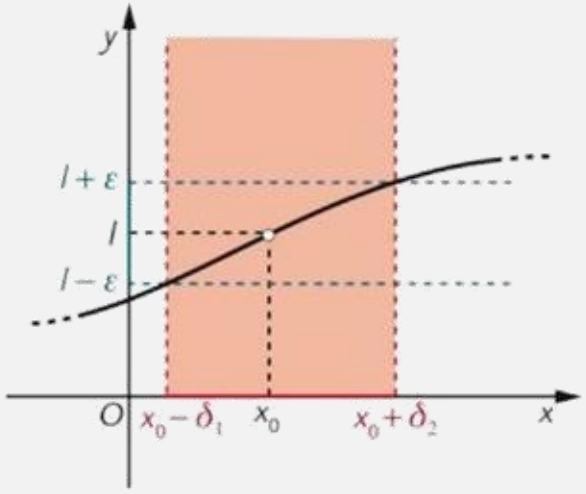
Let $f(x)$ be a function defined in an interval (a, b) excluding the point x_0.

We say that $f(x)$ tends to the limit L as x tends to x_0 if, for every arbitrary positive number ε, one can determine a neighborhood of x_0 such that for every x in that neighborhood:

$$|f(x) - L| < \varepsilon \rightarrow L - \varepsilon < f(x) < L + \varepsilon$$

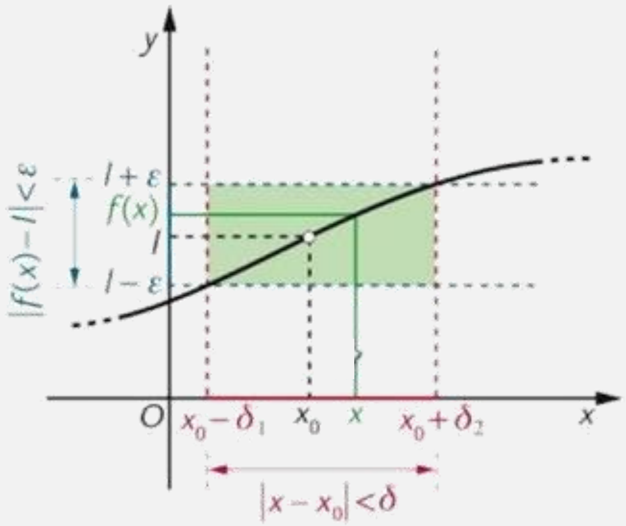


For every $\varepsilon > 0$ (by choosing ε, we select an arbitrary neighborhood of l on the y-axis)…



… there exists $\delta > 0$ (that identifies a proper neighborhood of x_0 on the x-axis)…

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \; \exists \; I_{x0} : \; \forall x \in I_{x0} - \{x_0\} \Longrightarrow |f(x) - L| < \varepsilon$$



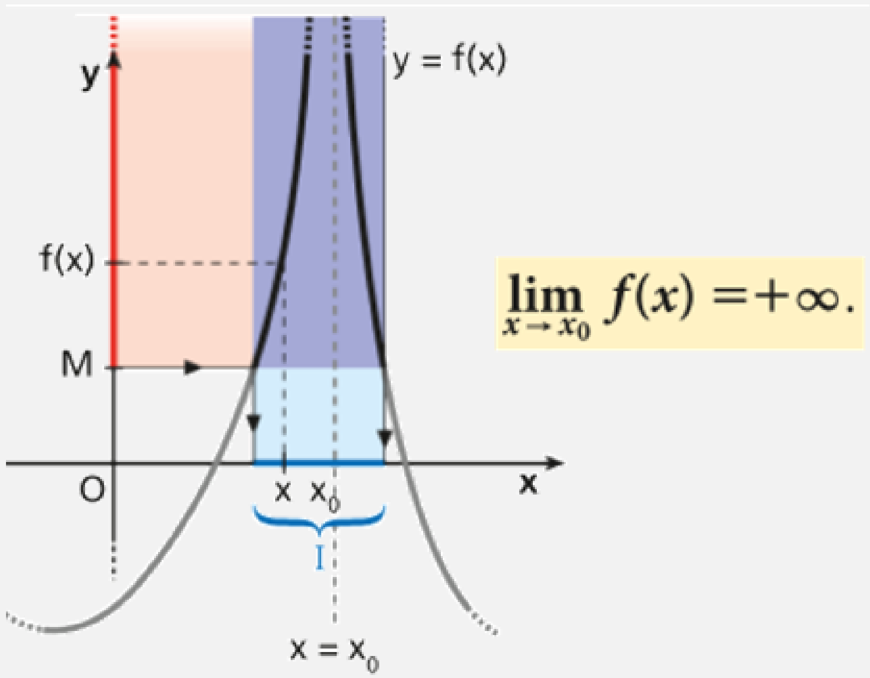
such that for each $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$ we have: $f(x) \in (l - \varepsilon, l + \varepsilon)$

➢ Infinite limit at a finite point:

$$L = \infty, x_0 \in \mathbb{R} \qquad \lim_{x \to x_0} f(x) = \infty$$

Let $f(x)$ be a function defined on an interval (a, b), excluding the point x_0.
We say that the function $f(x)$ tends to infinity as x approaches x_0 if, for every arbitrary positive number M, it is always possible to determine a (punctured) neighborhood of the point x_0 such that, for every x in that neighborhood, the following inequality holds:

$$|f(x)| > M$$



$$\lim_{x \to x_0} f(x) = +\infty.$$

$$\lim_{x \to x_0} f(x) = \infty \Leftrightarrow \forall n > 0 \; \exists \; I_{x0} : \; \forall x \in I_{x0} - \{x_0\} \Longrightarrow |f(x) - L| > M$$

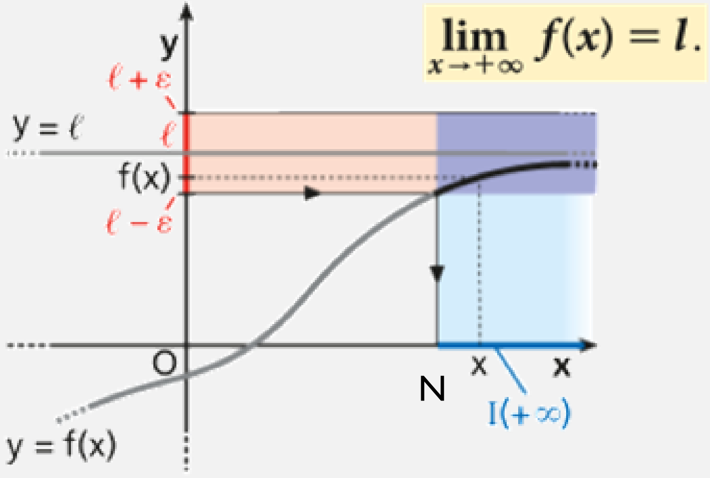
➢ **Finite limit at infinity**:

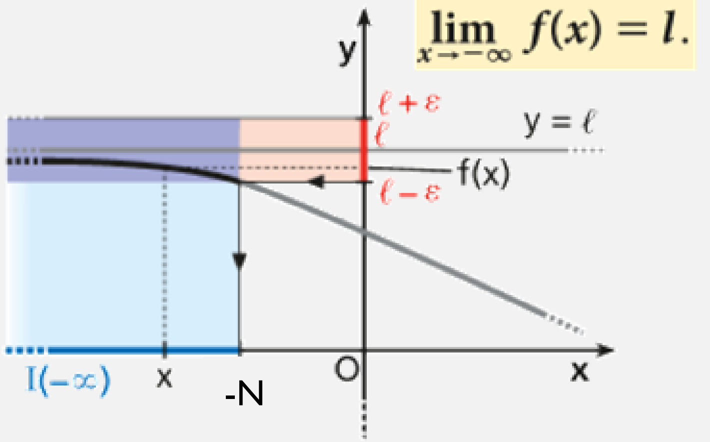
$L \in \mathbb{R}, x_0 = \infty \quad \lim_{x \to \infty} f(x) = L$

Let $f(x)$ be a function defined on an interval (a, b). We say that $f(x)$ tends to the limit L as x tends to infinity if, for every arbitrary positive number ε, it is always possible to determine a number $N > 0$ such that, for every x satisfying $|x| > N$, the following condition holds:

$$|f(x) - L| < \varepsilon \rightarrow L - \varepsilon < f(x) < L + \varepsilon$$

$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \; \exists \, N > 0 : \; \forall x : |x| > N \Longrightarrow |f(x) - L| < \varepsilon$$





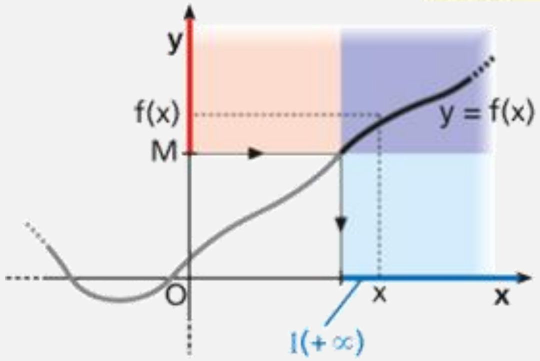
➢ **Infinite limit at infinity**:

$L, x_0 = \infty \quad \lim_{x \to \infty} f(x) = \infty$

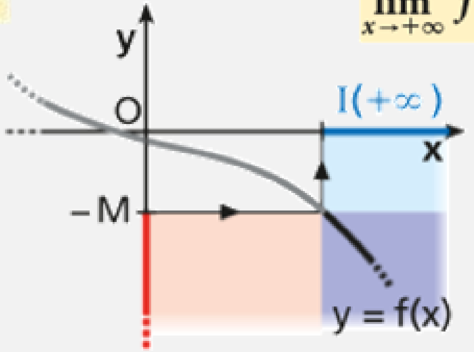
Let $f(x)$ be a function defined on an interval (a, b).
We say that $f(x)$ tends to infinity as x tends to infinity if, for every arbitrary positive number M, it is always possible to determine a number $N > 0$ such that, for every x satisfying $|x| > N$, the following inequality holds:

$$|f(x)| > M$$

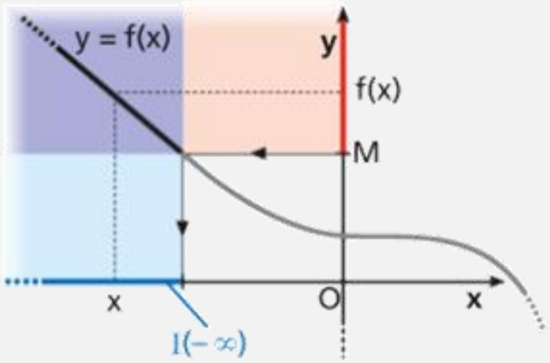
$\lim_{x \to +\infty} f(x) = +\infty.$



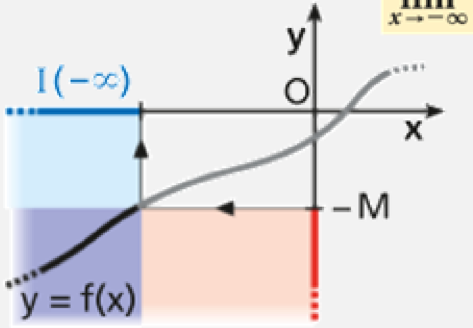
$\lim_{x \to +\infty} f(x) = -\infty.$



$\lim_{x \to -\infty} f(x) = +\infty.$



$\lim_{x \to -\infty} f(x) = -\infty.$



$$\lim_{x \to \infty} f(x) = \infty \Leftrightarrow \forall M > 0 \; \exists \, N > 0 : \; \forall x : |x| > N \Longrightarrow |f(x)| > M$$

Theorem: Uniqueness of the limit

A function cannot have two distinct limits at the same point; if:

$$\lim_{x \to x_0} f(x) = l \ \& \lim_{x \to x_0} f(x) = l' \to l = l'$$

Proof:

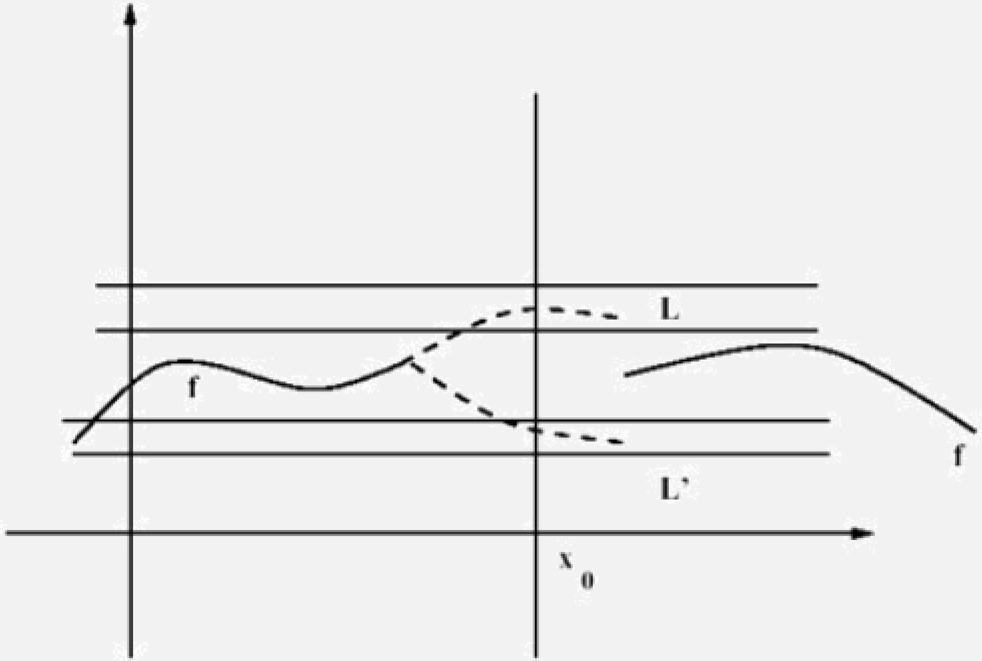
Assume $l \neq l'$. Then we can choose two disjoint neighborhoods of l and l', respectively:

$I_1 = I(l, \varepsilon_1), I_2 = I(l', \varepsilon_2), I_1 \cap I_2 = \emptyset.$

$f(x)$ with $x \in I(x_0, \delta)$: to which neighborhood would it belong?

One arrives to a contradiction: analytically, there exists δ_1 such that for $x \in D_f$:

$$0 < |x - x_0| < \delta_1, f(x) \in I_1$$



Theorem: Uniqueness of the limit

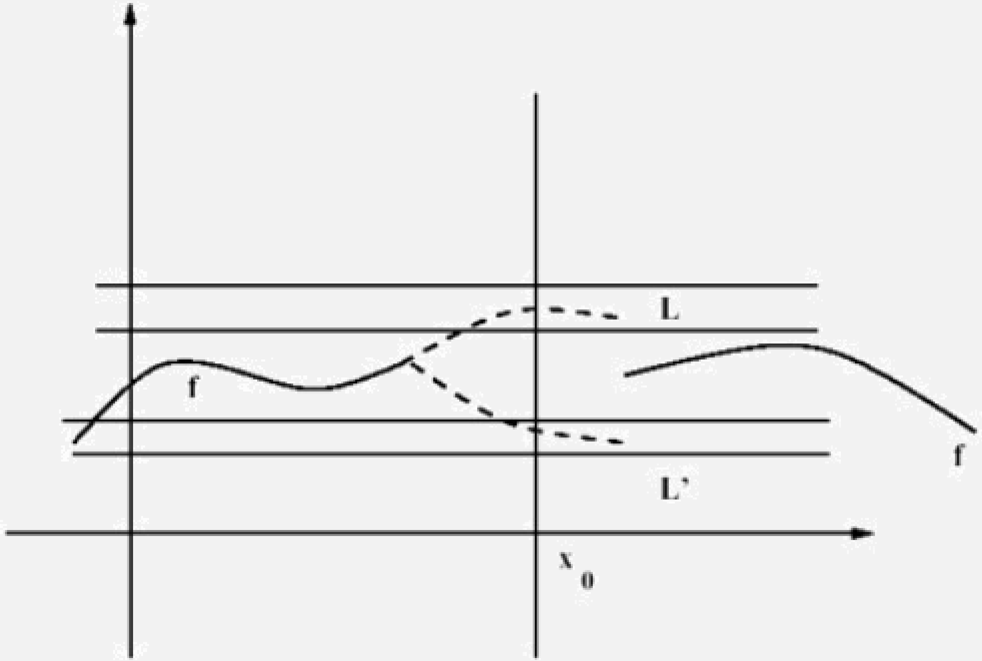
Similarly, there exists δ_2 such that for $x \in D_f$:

$$0 < |x - x_0| < \delta_2, f(x) \in I_2$$

Therefore, for $x \in D_f$:

$$0 < |x - x_0| < \min\{\delta_1, \delta_2\}, f(x) \in I_1 \; \& \; f(x) \in I_2$$

But this is impossible, since $I_1 \cap I_2 = \emptyset$



Right-hand and left-hand limits (extended real line)

- Let $x_0 = +\infty$: we call open neighborhood of x_0 an open half-line $(a, +\infty)$ and close neighborhood $[a, +\infty)$
- Let $x_0 = -\infty$: we call open neighborhood of x_0 an open half-line $(-\infty, a)$ and close neighborhood $(-\infty, a]$
- Let $a \subset \mathbb{R}$, $+\infty$ is an accumulation point of A if
$$A \cap (a, +\infty) \neq \emptyset \quad \forall a > 0$$
- Let $a \subset \mathbb{R}$, $\pm\infty$ is an accumulation point of A if
$$A \cap (-\infty, a) \neq \emptyset \quad \forall a < 0$$

Unified definition of Limit

Let $x_0, L \in \overline{\mathbb{R}}$.
Let $f : D_f \rightarrow \mathbb{R}$.
Let x_0 accumulation point for D_f.
$f(x)$ tends to L as x tends to x_0 → $\lim_{x \to x_0} f(x) = L$, if, for every neighborhood J of L there exists a neighborhood I of x_0 such that:

$$\forall x \in \left(I \cap D_f\right) - \{x_0\} \text{ one has that: } f(x) \in J$$

Right-hand limit

Let $x_0 \in \mathbb{R}$ and let $f : D_f \to \mathbb{R}$, assume that x_0 is an accumulation point of the set $D_f \cap (x_0, +\infty)$.

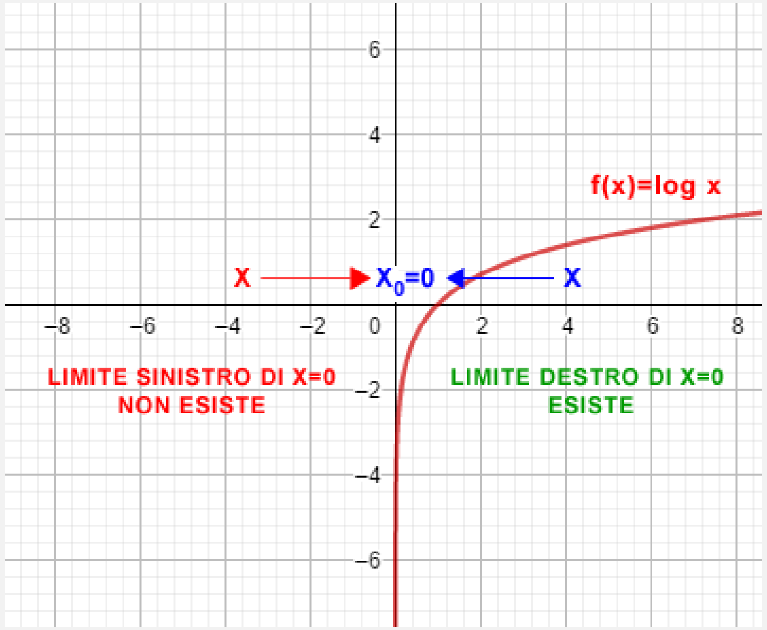
If the limit of the restriction of f to $D_f \cap (x_0, +\infty)$ exists as $x \to x_0$, then this value is called the **right-hand limit** of f at x_0, denoted by: :

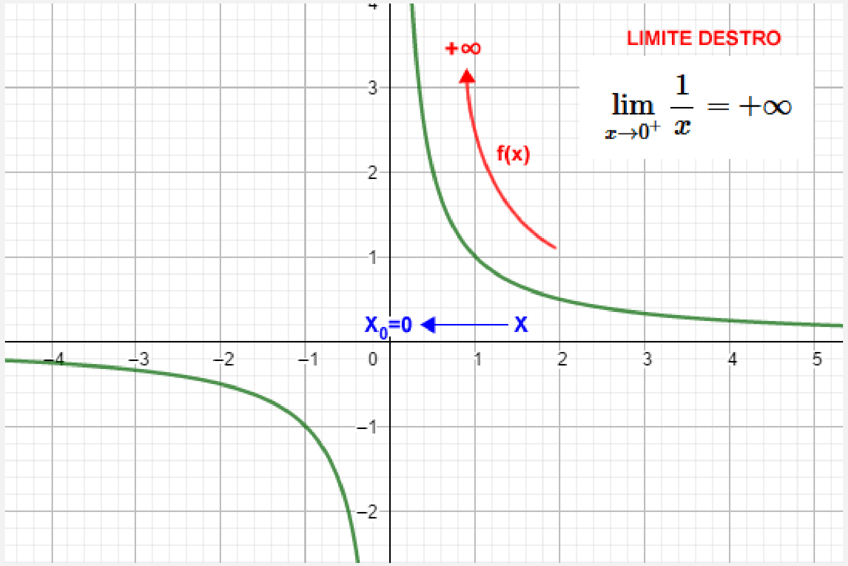
$$\lim_{x \to x_0^+} f(x)$$

Quantifier formulation (for $L \in \mathbb{R}$):

$$\lim_{x \to x_0^+} f(x) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in (x_0, x_0 + \delta) \cap D_f \;\; \to \quad |f(x) - L| \leq \varepsilon$$

$$x_0 < x < x_0 + \delta$$





Left-hand limit

Let $x_0 \in \mathbb{R}$ and let $f : D_f \to \mathbb{R}$, assume that x_0 is an accumulation point of the set $D_f \cap (-\infty, x_0)$.

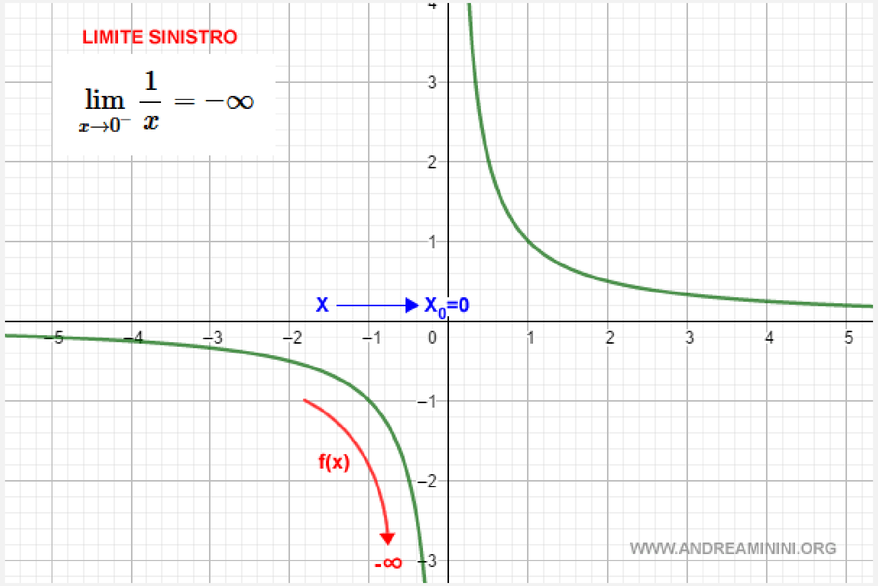
If the limit of the restriction of f to $D_f \cap (-\infty, x_0)$ exists as $x \to x_0$, then this value is called the **left-hand limit** of f at x_0, denoted by:

$$\lim_{x \to x_0^-} f(x)$$

Quantifier formulation (for $L \in \mathbb{R}$):

$$\lim_{x \to x_0^-} f(x) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in (x_0, x_0 + \delta) \cap D_f \;\; \to \;\;\; |f(x) - L| \leq \varepsilon$$

$$x_0 - \delta < x < x_0$$



Limits of polynomial functions

From the limit laws, it follows that the limit as $x \to x_0 \in \mathbb{R}$ exists for every polynomial function:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Using the definition of limit, it is easy to verify that:

$$\lim_{x \to x_0} a_0 = a_0, \qquad \lim_{x \to x_0} a_1 x = a_1 x_0, \qquad \lim_{x \to x_0} a_n x^n = a_n x_0$$

Therefore, in general:

$$\lim_{x \to x_0} p(x) = p(x_0) = a_0 + a_1 x_0 + \cdots + a_n x_0^n$$

Limits of quotients of polynomials (rational functions)

$$\lim_{x\to\pm\infty}\frac{P_n(x)}{Q_m(x)} = \begin{matrix} 1.\, n > m \to \lim = \pm\infty \\ 2.\, n < m \to \lim = 0 \\ 3.\, n = m \to \lim = a_n/b_m \end{matrix}$$

Examples.

1. $\lim_{x\to-\infty}\frac{1-x^2+3x^3}{2x^2-1+5x^3} = \frac{3}{5} \to n = m$

2. $\lim_{x\to-\infty}\frac{5x^3+2}{12x^4-x+7} = 0^- \;\; \to n < m$

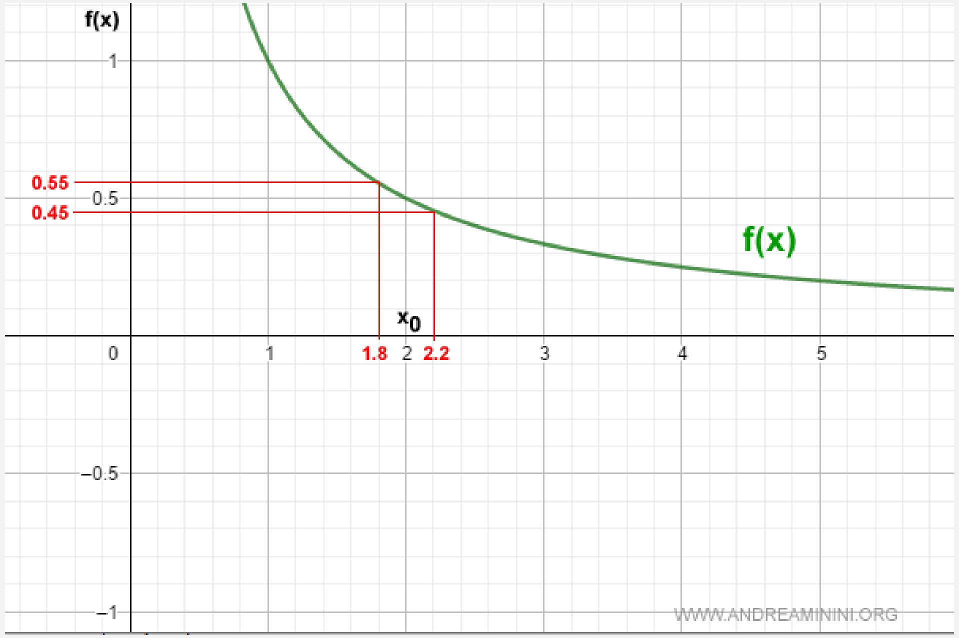
3. $\lim_{x\to 1^\pm}\frac{x^3-7x+1}{x^2-1} = \frac{-3}{0} = \pm\infty \;\; \to n > m$

Theorem: Sign preservation

Let $x_0 \in \mathbb{R}$. If $\lim_{x \to x_0} f(x) = L > 0$, then there exists a neighborhood I of x_0 such that:

$$f(x) > 0 \ \forall x \in I - \{x_0\} \cap D_f$$

If a function f is defined and continuous in a neighborhood of x_0 and $f(x_0) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for every x in the $(x_0 - \delta, x_0 + \delta)$ neighborhood



Example. Let $f(x) = \frac{1}{x}$.
In $x_0 = 2$ the funtion $f(x_0) = 0.5 \rightarrow f(2) = 0.5$

We consider a neighborhood of $x_0 = 2$ with $\delta = 0.2$:

$$(x_0 - \delta, x_0 + \delta) = (2 - 0.2, 2 + 0.2) = (1.8, 2.2)$$

In the neighborhood $(1.8, 2.2)$ the function is always > 0:

$$f(1.8) = \frac{1}{1.8} = 0.55$$

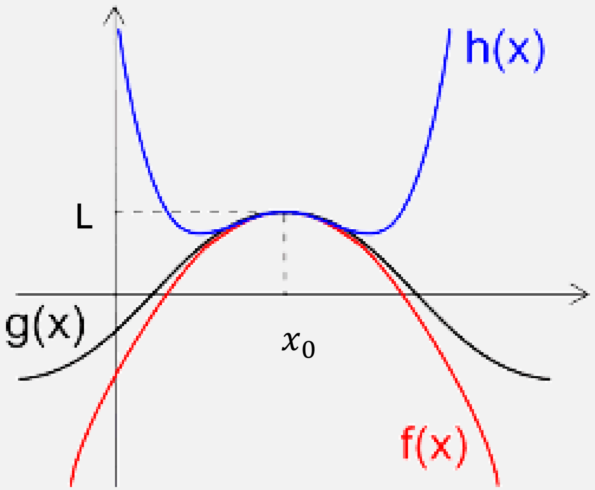
$$f(2.2) = \frac{1}{2.2} = 0.45$$

Squeeze theorem (comparison theorem for limits)

Let $A \subseteq \mathbb{R}$, and $f, g, h : A \to \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be an accumulation point of A and suppose that in a neighborhood I of the point x_0 we have:

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in I - \{x_0\}$$

If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then also g adits the limit as $x \to x_0$ and we have $\lim_{x \to x_0} g(x) = L$



Let $\varepsilon > 0$, from $\lim_{x \to x_0} f(x) = L$ it derives that $\exists \delta_1 > 0 : |f(x) - L| < \varepsilon$ for $0 < |x - x_0| < \delta_1$

Similarly, $\exists \delta_2 > 0 : |h(x) - L| < \varepsilon$ for $0 < |x - x_0| < \delta_2$

Once selected $\bar{\delta} = \min\{\delta_1, \delta_2\}$, for $0 < |x - x_0| < \bar{\delta}$, then: $L - \varepsilon < h(x) < L + \varepsilon \;\&\; L - \varepsilon < f(x) < L + \varepsilon$

Hence, for $0 < |x - x_0| < \min\{\bar{\delta}, r\} = \delta \;\; \to L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$, from which:

$$|g(x) - L| < \varepsilon \;\; \to \;\; \lim_{x \to x_0} g(x) = L$$

Recall: monotone functions

A function $f : D_f \to \mathbb{R}$ is called:

- Non-decreasing (**monotone increasing**) if

$$\forall x, y \in D_f : x \leq y \implies f(x) \leq f(y)$$

- Non-increasing (**monotone decreasing**) if

$$\forall x, y \in D_f : x \leq y \implies f(x) \geq f(y)$$

- Strictly increasing if

$$\forall x, y \in D_f : x < y \implies f(x) < f(y)$$

- Strictly decreasing if

$$\forall x, y \in D_f : x < y \implies f(x) > f(y)$$

Theorem: limits of monotone functions

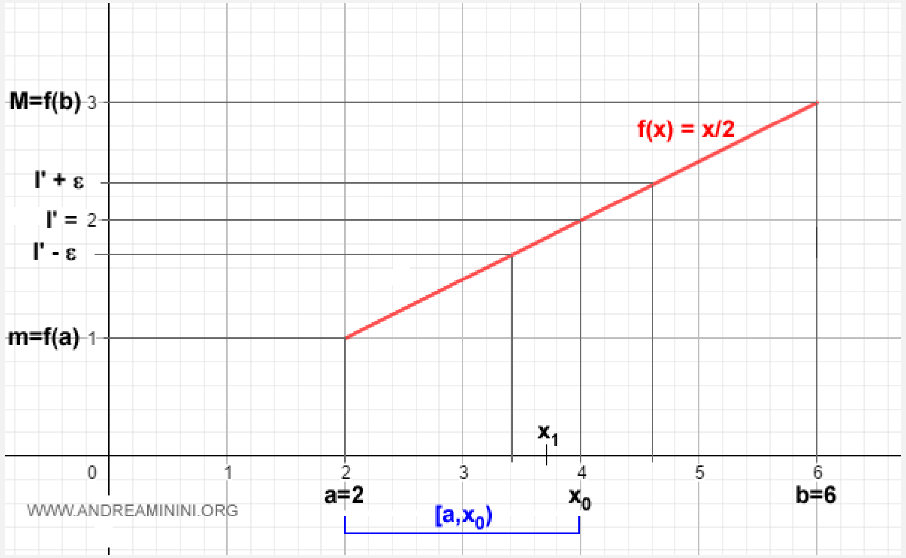
Let $f : D_f \to \mathbb{R}$ be an increasing function and let $x_0 \in \mathbb{R}$ be an accumulation point for D_f.
Then, the one-sided limits exist and:

$$\exists \lim_{x \to x_0^+} f(x), \quad \exists \lim_{x \to x_0^-} f(x)$$

$$\lim_{x \to x_0^-} f(x) = \sup\{f(x) : x \in D_f, x < x_0\}$$

$$\lim_{x \to x_0^+} f(x) = \inf\{f(x) : x \in D_f, x > x_0\}$$

Analogous results hold for decreasing functions.



Infinitesimals and infinities

Let f be a real-valued function defined on an interval I, possibly excluding a point $x_0 \in I$:

- f is **infinitesimal** as $x \to x_0$ or $x \to \pm\infty$ if

$$\lim_{x \to x_0} f(x) = 0 \quad \text{or} \quad \lim_{x \to \pm\infty} f(x) = 0$$

- f is **infinite** as $x \to x_0$ or $x \to \pm\infty$ if

$$\lim_{x \to x_0} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to \pm\infty} f(x) = \pm\infty$$

Indeterminate forms

Indeterminate forms arise from operations involving infinitesimals and infinities for which the limit cannot be determined a priori:

$$\infty - \infty \quad 0 \cdot \infty \quad \frac{\infty}{\infty} \quad \frac{0}{0} \quad 0^0 \quad 1^\infty \quad \infty^0$$

Non-indeterminate forms include:

$$k + \infty = +\infty$$
$$k - \infty = -\infty$$
$$\infty + \infty = \infty$$
$$k \cdot \infty = \infty$$
$$\infty \cdot \infty = \infty$$
$$\frac{k}{\infty} = 0$$
$$\frac{0}{\infty} = 0$$
$$\frac{\infty}{k} = \infty$$
$$\frac{k}{0} = \infty$$

Indeterminate forms – algebraic methods

To resolve indeterminate forms, one may:

$\lim\limits_{x\to\pm\infty} \text{polynomial} = +\infty - \infty$	Factor out the highest-degree power of x
$\lim\limits_{x\to\pm\infty} \dfrac{\text{polynomial}}{\text{polynomial}} = \dfrac{\pm\infty}{\pm\infty}$	Factor out the highest-degree power of x at both numerator and denominator; Simplify numerator and denominator
$\lim\limits_{x\to x_0} \dfrac{\text{polynomial}}{\text{polynomial}} = \dfrac{0}{0}$	Factor polynomials and cancel common terms

Indeterminate forms – algebraic methods

To resolve indeterminate forms, one may:

$$\lim_{x\to+\infty} \sqrt{A} - \sqrt{B} = +\infty - \infty$$

Recalling that: $\left(\sqrt{A}-\sqrt{B}\right)\left(\sqrt{A}+\sqrt{B}\right) = A - B,$

Multiply and divide by the conjugate expression $\sqrt{A}+\sqrt{B}$

$$\lim_{x\to+\infty} \sqrt[3]{A} - \sqrt[3]{B} = +\infty - \infty$$

Recalling that: $\left(\sqrt[3]{A}-\sqrt[3]{B}\right)\left[\left(\sqrt[3]{A}\right)^2+\sqrt[3]{AB}+\left(\sqrt[3]{B}\right)^2\right] = A - B,$

Multiply and divide by $\left(\sqrt[3]{A}\right)^2+\sqrt[3]{AB}+\left(\sqrt[3]{B}\right)^2$

Indeterminate forms – algebraic methods

Faster alternative method to the first two cases:

$$\lim_{x \to \pm\infty} \text{polynomial} = +\infty - \infty$$

Substitute $+\infty$ or $-\infty$ for the highest-degree term and neglect lower-order terms.

$$\lim_{x \to \pm\infty} \frac{\text{polynomial}}{\text{polynomial}} = \frac{\pm\infty}{\pm\infty}$$

If the numerator has higher degree → limit is $\pm\infty$, depending on the signs.

If degrees are equal → ratio of leading coefficients.

If denominator has higher degree → limit is zero.

Examples

$$\lim_{x\to\pm\infty} polinomio = +\infty - \infty$$

$$\lim_{x\to+\infty} 2x^3 - x^2 + 3 = 2\cdot(+\infty)^3 - (+\infty)^2 + 3 = 2\cdot(+\infty) - (+\infty) + 3 = +\infty - \infty$$

la forma indeterminata $+\infty - \infty$ si risolve raccogliendo la x di grado massimo del polinomio, cioè:

$$\lim_{x\to+\infty} 2x^3 - x^2 + 3 = \lim_{x\to+\infty} x^3\left(2 - \frac{1}{x} + \frac{3}{x^3}\right) = (+\infty)^3\cdot\left(2 - \frac{1}{+\infty} + \frac{3}{(+\infty)^3}\right) =$$

$$= +\infty\cdot(2 - 0 + 0) = +\infty\cdot 2 = +\infty$$

Soluzione alternativa più rapida:

$$\lim_{x\to+\infty} 2x^3 - x^2 + 3 = \lim_{x\to+\infty} 2x^3 = 2\cdot(+\infty)^3 = 2\cdot(+\infty) = +\infty$$

$$\lim_{x\to-\infty} 2x^3 - x^2 + 3 = \lim_{x\to-\infty} 2x^3 = 2\cdot(-\infty)^3 = 2\cdot(-\infty) = -\infty$$

Examples

$$\lim_{x\to\pm\infty} \frac{polinomio}{polinomio} = \frac{\pm\infty}{\pm\infty}$$

$$\lim_{x\to+\infty} \frac{x+2}{2x^2-5} = \frac{(+\infty)+2}{2\cdot(+\infty)^2-5} = \frac{+\infty}{2\cdot(+\infty)-5} = \frac{+\infty}{+\infty}$$

la forma indeterminata $\frac{+\infty}{+\infty}$ si risolve raccogliendo la x di grado massimo al numeratore e al denominatore:

$$\lim_{x\to+\infty} \frac{x+2}{2x^2-5} = \lim_{x\to+\infty} \frac{x\cdot\left(1+\frac{2}{x}\right)}{x^2\cdot\left(2-\frac{5}{x^2}\right)} = \lim_{x\to+\infty} \frac{\left(1+\frac{2}{x}\right)}{x\cdot\left(2-\frac{5}{x^2}\right)} = \frac{1+\frac{2}{(+\infty)}}{(+\infty)\cdot\left(2-\frac{5}{(+\infty)^2}\right)} = \frac{1}{(+\infty)\cdot 2} = \frac{1}{(+\infty)} = 0$$

Examples

$$\lim_{x\to\pm\infty} \frac{polinomio}{polinomio} = \frac{\pm\infty}{\pm\infty}$$

numerator has higher degree

$$\lim_{x\to+\infty} \frac{4x^5 - 3x + 2}{2x^2 + 4} = +\infty \qquad \lim_{x\to+\infty} \frac{-3x^2 + 2x - 5}{x - 4} = -\infty$$

$$\lim_{x\to-\infty} \frac{4x^5 - 3x + 2}{2x^2 + 4} = -\infty \qquad \lim_{x\to-\infty} \frac{2x^4 + 5x^3}{x^3 + 2x^2 - 1} = +\infty$$

degrees are equal

$$\lim_{x\to+\infty} \frac{7x^3 - 3x + 2}{2x^3 + 4} = \frac{7}{2} \qquad \lim_{x\to+\infty} \frac{3x^2 - x + 2}{4x - 2x^2 - 1} = -\frac{3}{2}$$

$$\lim_{x\to-\infty} \frac{7x^3 - 3x + 2}{2x^3 + 4} = \frac{7}{2} \qquad \lim_{x\to-\infty} \frac{3x^2 - x + 2}{4x - 2x^2 - 1} = -\frac{3}{2}$$

denominator has higher degree

$$\lim_{x\to+\infty} \frac{5x^3 - 3x + 1}{2x^4 + 4} = 0 \qquad \lim_{x\to+\infty} \frac{x^2 + 2x + 1}{2x^3 + 4} = 0$$

$$\lim_{x\to-\infty} \frac{5x^3 - 3x + 1}{2x^4 + 4} = 0 \qquad \lim_{x\to-\infty} \frac{2x + 1}{-2x^2 + 4x + 3} = 0$$

Examples

$$\lim_{x \to x_0} \frac{polinomio}{polinomio} = \frac{0}{0}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

la forma indeterminata $\frac{0}{0}$ si risolve scomponendo numeratore e denominatore e poi semplificando:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1) \cdot (x + 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Examples

$$\lim_{x\to+\infty} \sqrt{A}-\sqrt{B} = +\infty-\infty$$

$$\lim_{x\to+\infty}\left(\sqrt{3x+1}-\sqrt{x}\right) = \sqrt{3\cdot(+\infty)}-\sqrt{(+\infty)} = \sqrt{+\infty}-\sqrt{+\infty} = +\infty-\infty$$

$$\lim_{x\to+\infty}\left(\sqrt{3x+1}-\sqrt{x}\right) = \lim_{x\to+\infty}\left(\sqrt{3x+1}-\sqrt{x}\right)\cdot\frac{\left(\sqrt{3x+1}+\sqrt{x}\right)}{\sqrt{3x+1}+\sqrt{x}} = \lim_{x\to+\infty}\frac{(3x+1)-x}{\sqrt{3x+1}+\sqrt{x}} =$$

$$= \boldsymbol{\lim_{x\to+\infty}\frac{2x+1}{\sqrt{3x+1}+\sqrt{x}}} = \frac{2\cdot(+\infty)+1}{\sqrt{3\cdot(+\infty)+1}+\sqrt{(+\infty)}} = \frac{+\infty}{+\infty}$$

la forma indeterminata $\frac{+\infty}{+\infty}$ si risolve applicando la tecnica vista in precedenza, cioè:

$$\boldsymbol{\lim_{x\to+\infty}\frac{2x+1}{\sqrt{3x+1}+\sqrt{x}}} = \lim_{x\to+\infty}\frac{x\cdot\left(2+\frac{1}{x}\right)}{\sqrt{x}\cdot\left[\sqrt{3+\frac{1}{x}}+1\right]} = \boldsymbol{\lim_{x\to+\infty}\frac{\sqrt{x}\cdot\left(2+\frac{1}{x}\right)}{\sqrt{3+\frac{1}{x}}+1}} = \frac{\sqrt{+\infty}\cdot\left(2+\frac{1}{(+\infty)}\right)}{\sqrt{3+\frac{1}{(+\infty)}}+1} = \frac{+\infty}{\sqrt{3}+1} = +\infty$$

Limits of continuous functions

Let a real-valued function f defined on an interval I be continuous at $x_0 \in I$

f is **continuous** at x_0 if:

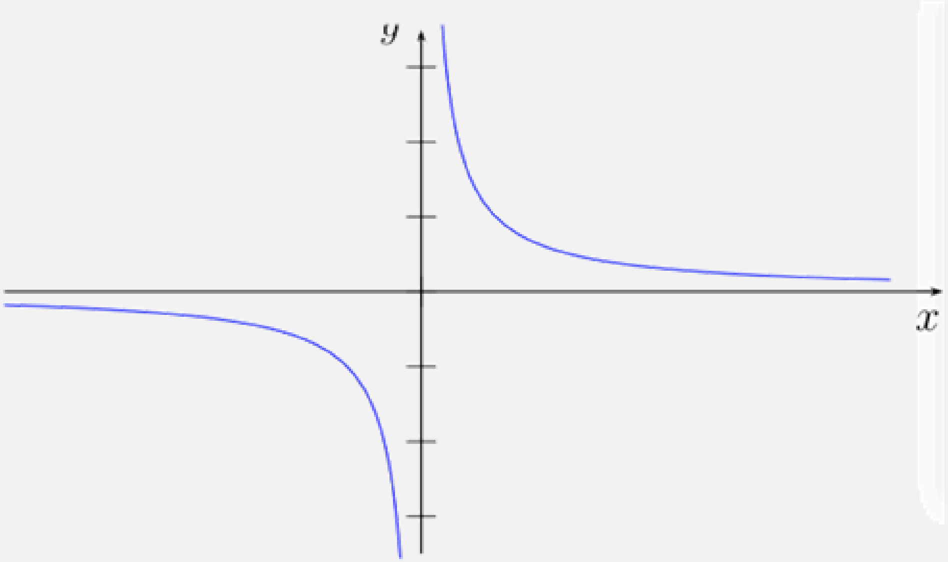
$$\lim_{x \to x_0} f(x) = f(x_0)$$

Hence, f is continuous in a point x_0 internal to the definition interval if:

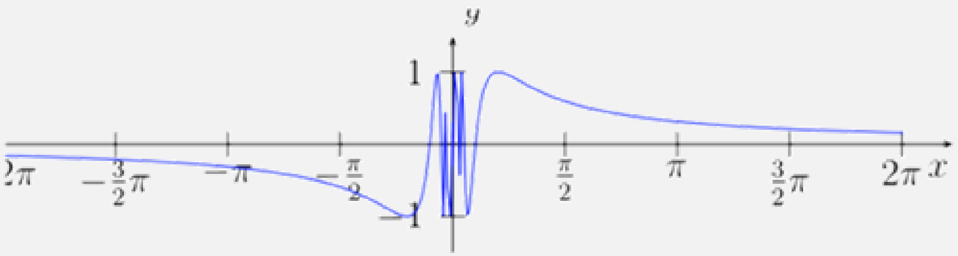
- $\exists f(x_0)$
- $\exists$ finite limit of $f(x)$ as $x \to x_0$
- the limit coincides with the value $f(x_0)$ of f at x_0

Limits of continuous functions

$f : x \mapsto \frac{1}{x}$ is continuous in $D_f = \mathbb{R} \setminus \{0\}$



$f : x \mapsto \sin\left(\frac{1}{x}\right)$ is continuous in $\operatorname{dom} f = \mathbb{R} \setminus \{0\}$



Limits of continuous functions

- If it holds only that:

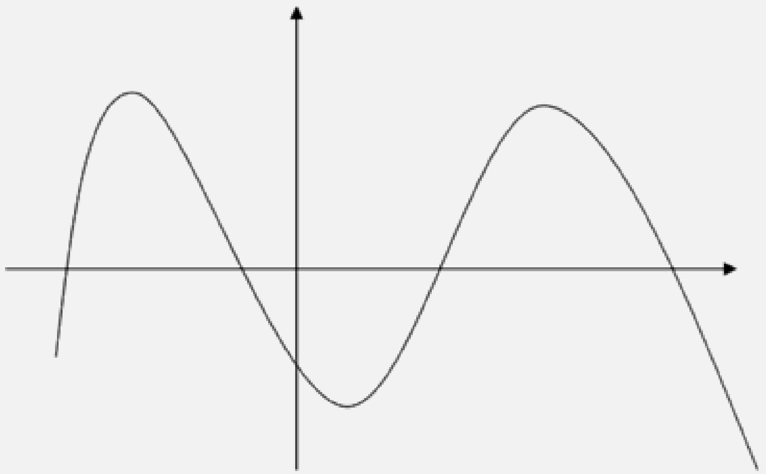
$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

then the function is said to be **right-continuous at x_0**

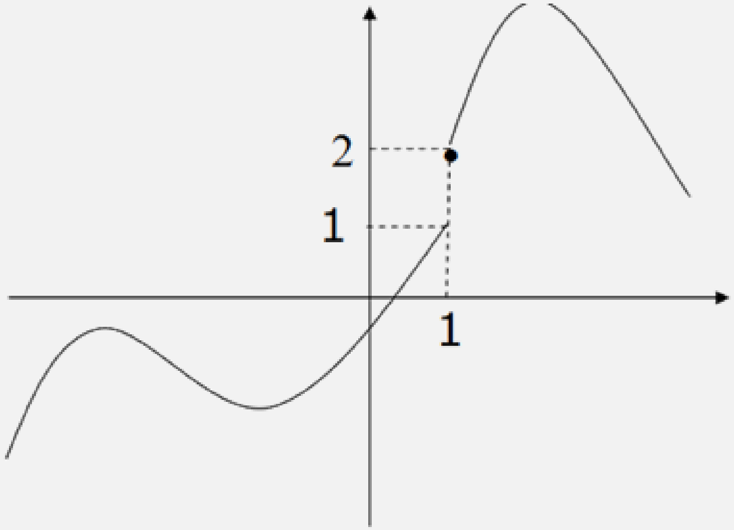
- If it holds only that :

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

then the function is said to be **left-continuous at x_0**



Continuous function



Non-continuous function

Limits of continuous functions

All elementary functions introduced so far are continuous on their respective domains.
The sum, product, quotient, and composition of continuous functions are also continuous functions.

$$\begin{aligned}
f(x) &= x^r, && r \in \mathbb{R},\\
f(x) &= a^x, && a \in \mathbb{R}^+ \setminus \{1\},\\
f(x) &= \log_a(x), && a \in \mathbb{R}^+ \setminus \{1\},\\
f(x) &= \sin(x),\\
f(x) &= \cos(x),\\
f(x) &= \tan(x)\\
f(x) &= \arcsin(x),\\
f(x) &= \arccos(x),\\
f(x) &= \arctan(x)
\end{aligned}$$

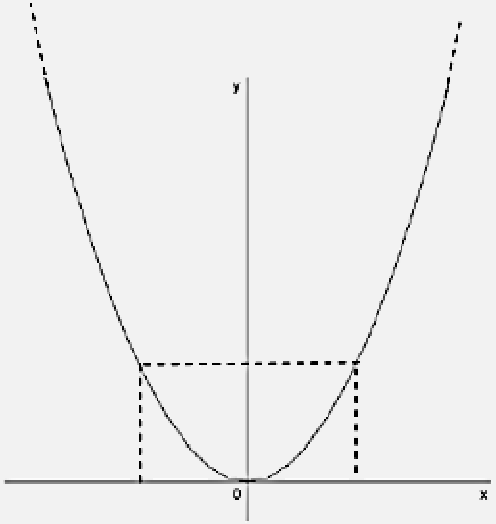
Limits of elementary functions

Power function with integer exponent

$$\lim_{x\to+\infty} x^n = +\infty,$$

$$\forall n \in \mathbb{N}$$

n even



n even

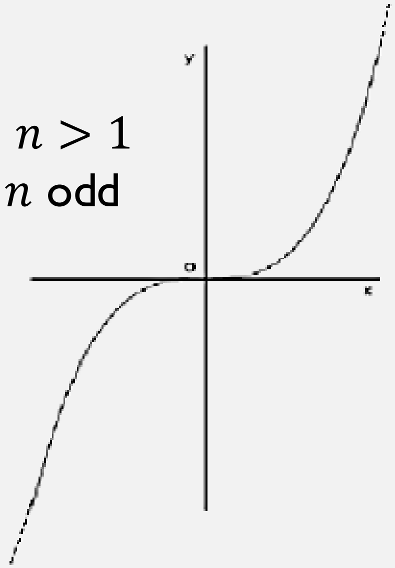
$$\lim_{x\to-\infty} x^n = +\infty$$

n even

$$\lim_{x\to 0} x^n = 0$$

$n > 1$

n odd



n odd

$$\lim_{x\to-\infty} x^n = -\infty$$

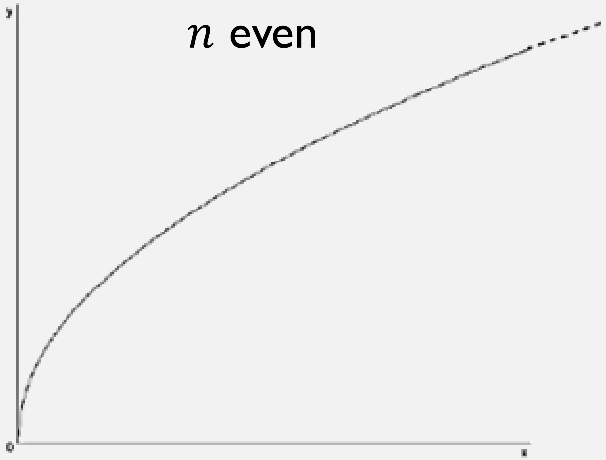
n odd

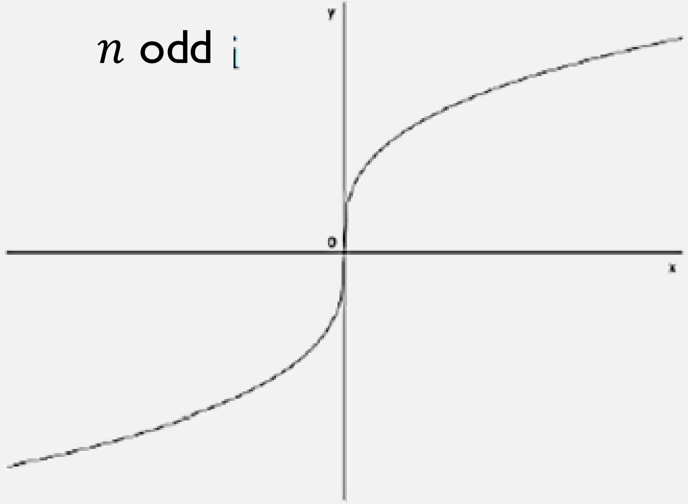
$$\lim_{x\to 0} x^n = 0$$

Limits of elementary functions

nth root function

$$\lim_{x \to +\infty} \sqrt[n]{x} = +\infty,$$
$$\forall n \in \mathbb{N}$$





n even

$$\nexists \lim_{x \to -\infty} \sqrt[n]{x}$$

n odd

$$\lim_{x \to -\infty} \sqrt[n]{x} = -\infty$$

n even

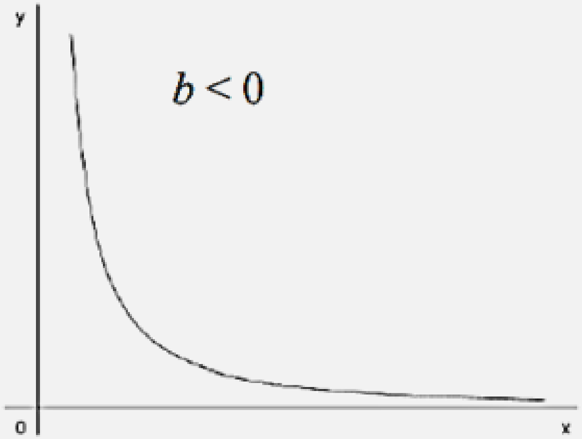
$$\lim_{x \to 0^+} \sqrt[n]{x} = 0^+$$

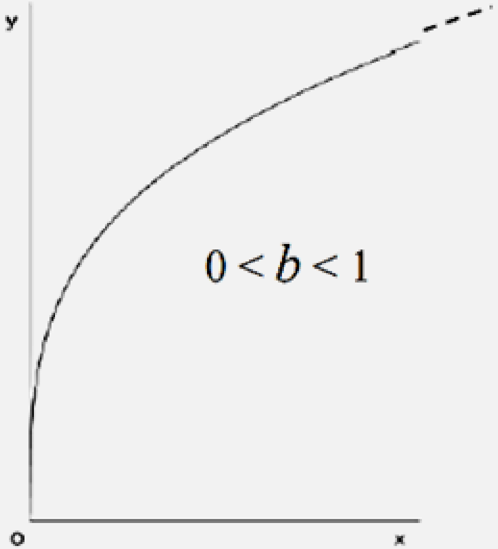
n odd

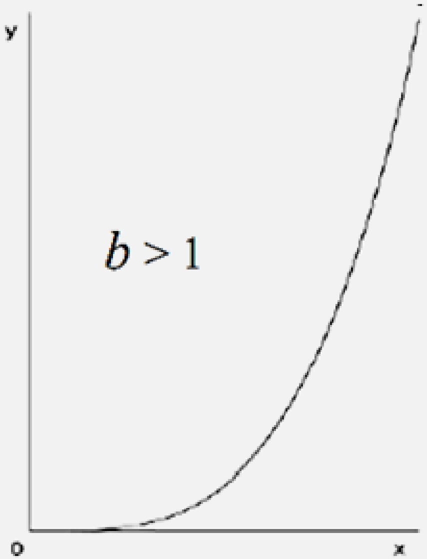
$$\lim_{x \to 0} \sqrt[n]{x} = 0$$

Limits of elementary functions

Power function with real exponent







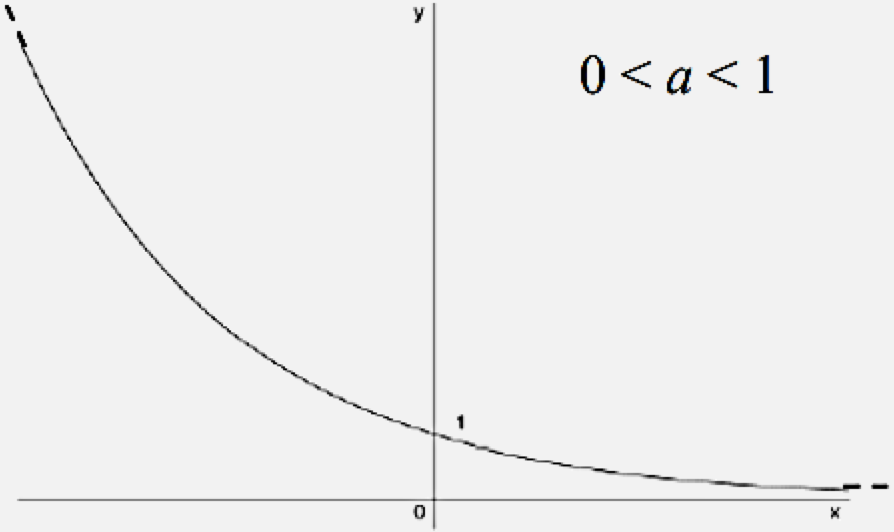
$$\lim_{x\to+\infty} x^b = 0 \qquad \lim_{x\to+\infty} x^b = +\infty \qquad \lim_{x\to+\infty} x^b = +\infty$$

$$\lim_{x\to0^+} x^b = +\infty \qquad \lim_{x\to0} x^b = 0 \qquad \lim_{x\to0} x^b = 0$$

Continuity at zero

Limits of elementary functions

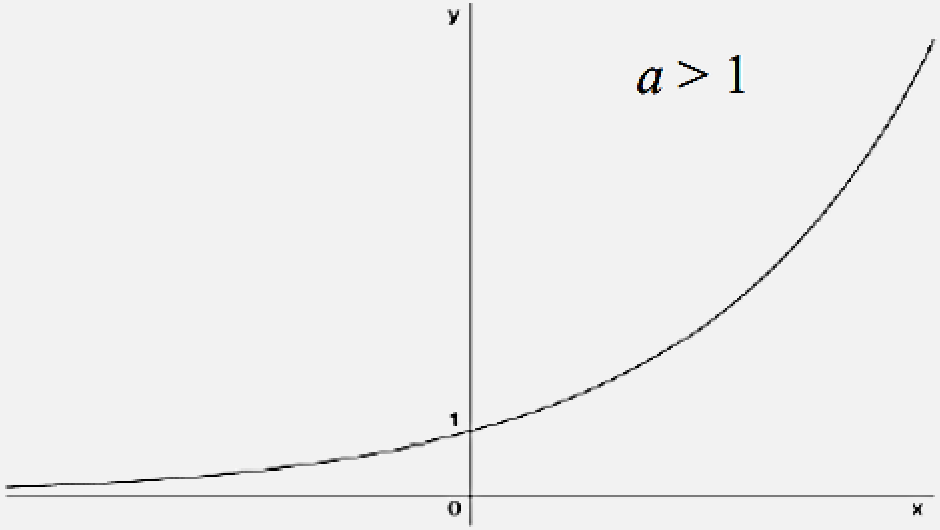
Exponential function



$0 < a < 1$

$$\lim_{x\to+\infty} a^x = 0$$

$$\lim_{x\to-\infty} a^x = +\infty$$



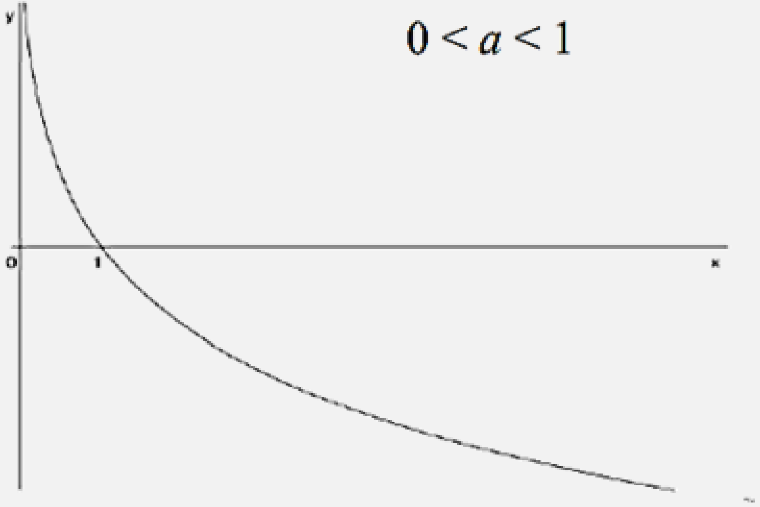
$a > 1$

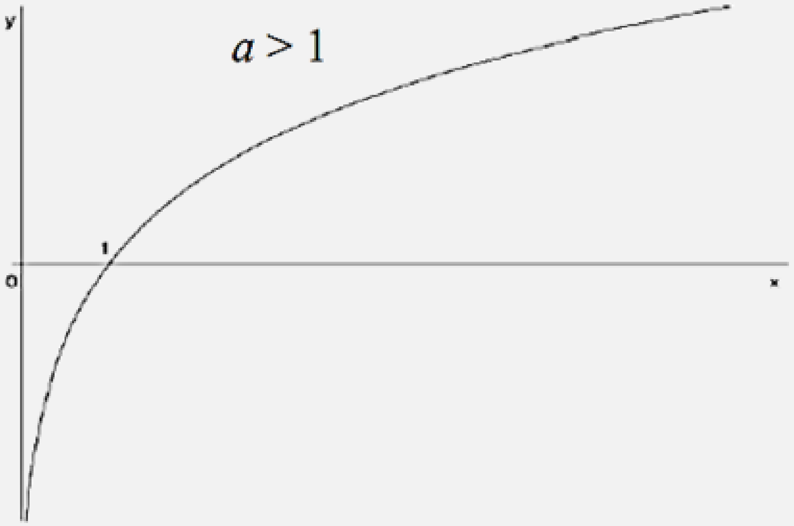
$$\lim_{x\to+\infty} a^x = +\infty$$

$$\lim_{x\to-\infty} a^x = 0$$

Limits of elementary functions

Logarithmic function





$$\lim_{x \to +\infty} \log_a x = -\infty$$

$$\lim_{x \to 0^+} \log_a x = +\infty$$

$$\lim_{x \to +\infty} \log_a x = +\infty$$

$$\lim_{x \to 0^+} \log_a x = -\infty$$

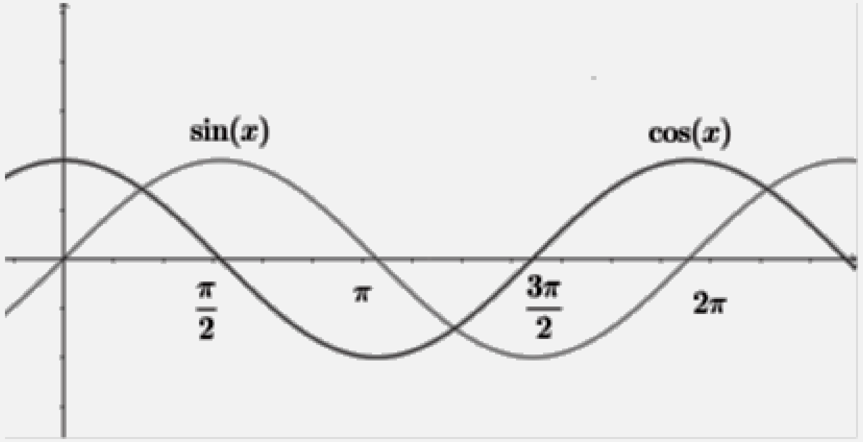
Limits of elementary functions

Sine and cosine functions

$$\nexists \lim_{x \to \pm\infty} \sin x$$

$$\nexists \lim_{x \to \pm\infty} \cos x$$

→ Although these limits do not exist, they represent **bounded (finite) behavior,** since both sine and cosine take values in the closed and bounded interval $[-1,1]$



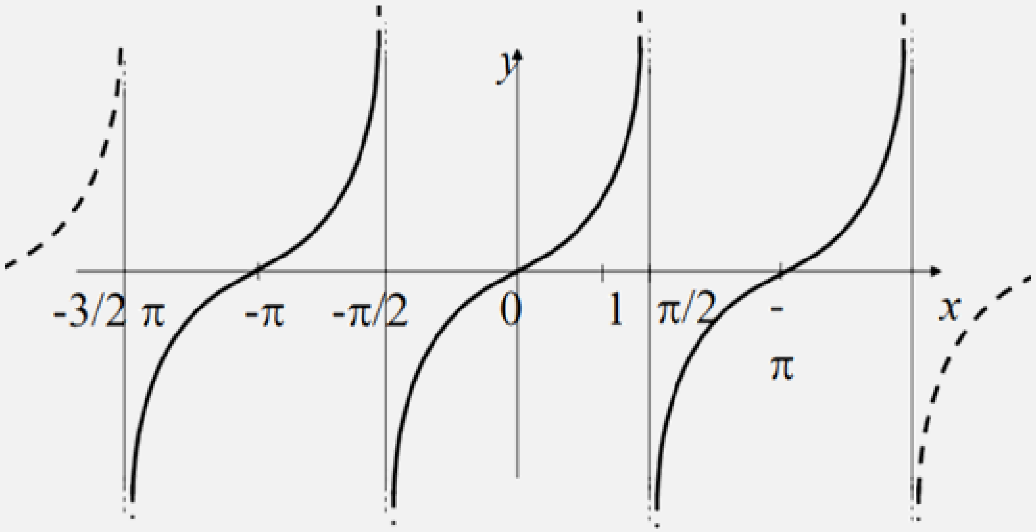
$$\lim_{x \to 0} \sin x = 0 \qquad \lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to \frac{\pi}{2}} \sin x = 1 \qquad \lim_{x \to \frac{\pi}{2}} \cos x = 0$$

Limits of elementary functions

Tangent function

$$\nexists \lim_{x \to \pm\infty} \tan x$$



$$\lim_{x \to 0} \tan x = 0$$

$$\lim_{x \to \frac{\pi}{2}^{+}} \tan x = -\infty$$

$$\lim_{x \to \frac{\pi}{2}^{-}} \tan x = +\infty$$

Horizontal asymptotes

Given a real-valued function f defined on an unbounded interval I, if the limit at infinity is finite, that is,:

$$\lim_{x\to\pm\infty} f(x) = l$$

then, the line $y = l$ is called **horizontal asymptote** for the graph of $f(x)$

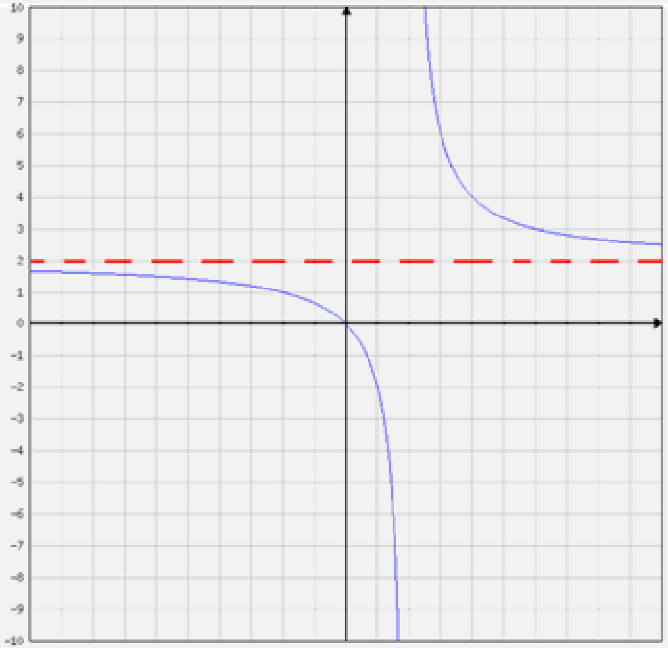
$$\lim_{x\to-\infty} f(x) = 2$$

$$\lim_{x\to+\infty} f(x) = 2$$

$$\Downarrow$$

$$y = 2$$

horizontal asymptote



Horizontal asymptotes

If

$$\lim_{x\to+\infty} f(x) = l_1 \qquad \lim_{x\to-\infty} f(x) = l_2$$

then the graph of $f(x)$ admits two horizontal asymptotes: $y = l_1$ as $x \to +\infty$ and $y = l_2$ as $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = -2$$

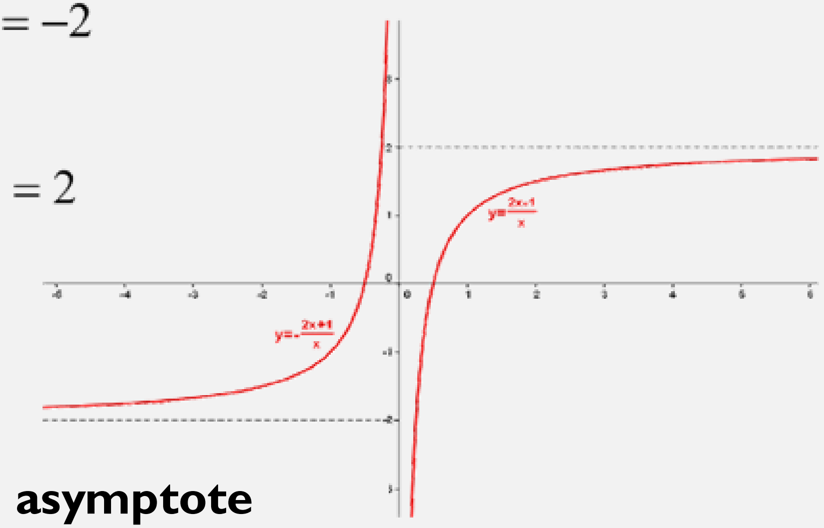
$$\lim_{x\to+\infty} f(x) = 2$$

$$y = 2$$

right horizontal asymptote

$$y = -2$$

left horizontal asymptote



Horizontal asymptotes

If the limit at infinity of a real-valued function defined on an unbounded interval is infinite, that is,

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

then the function does not admit a horizontal asymptote.

Oblique asymptotes

Given a real-valued function f defined on an unbounded interval I, if

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

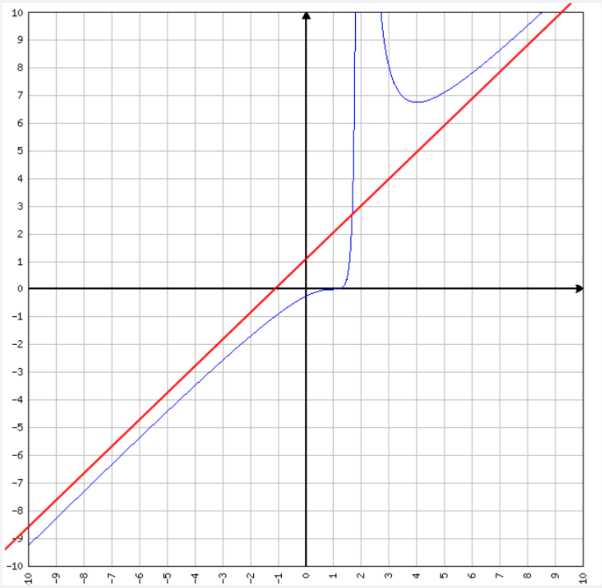
then the function DOES NOT admit a horizontal asymptote.
It is possible to check whether the graph admits an oblique asymptote of equation $y = mx + q$.

The graph of $f(x)$ admits an oblique asymptote if:

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = m \neq 0 \; finite$$

$$\lim_{x\to\pm\infty} [f(x) - mx] = q \; finite$$



Oblique asymptotes

If:

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = m = 0 \text{ or infinite}$$

$$\lim_{x \to \pm\infty} [f(x) - mx] = q \text{ infinite}$$

→ then the graph of $f(x)$ does not admit an oblique asymptote

Vertical asymptotes

Given a real-valued function f defined on an interval I missing at most the point x_0 , if the limit at the finite point x_0 is infinite, that is,

$$\lim_{x\to x_0^+} f(x) = \pm\infty$$

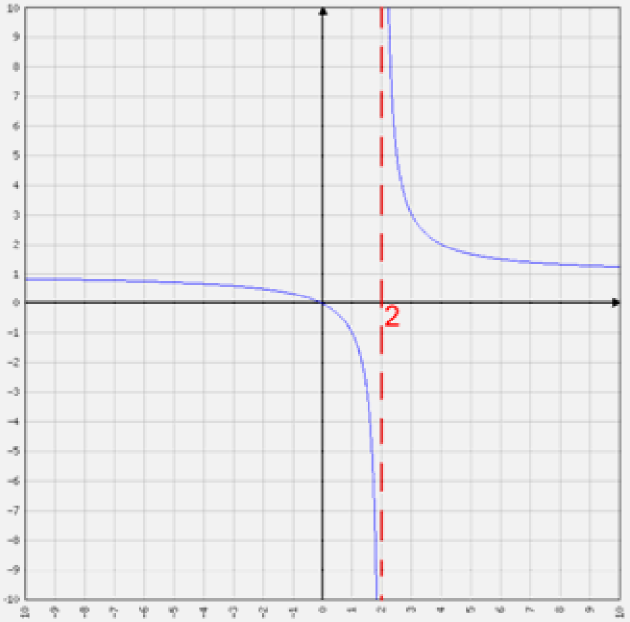
then the line $x = x_0$ is called a **vertical asymptote** of the graph of $f(x)$

$$\lim_{x\to 2^+} f(x) = +\infty$$

$$\lim_{x\to 2^-} f(x) = -\infty$$

$$x = 2$$

vertical asymptote

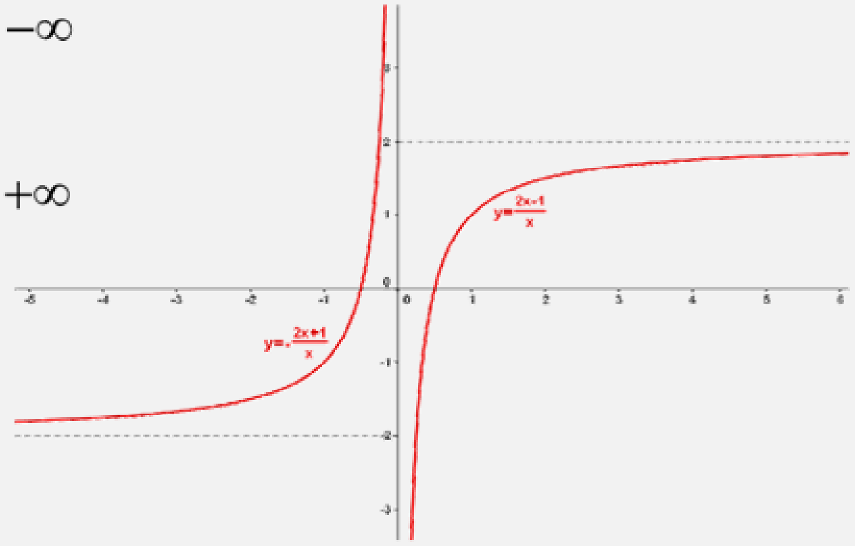


$$\lim_{x\to 0^+} f(x) = -\infty$$

$$\lim_{x\to 0^-} f(x) = +\infty$$

$$x = 0$$

vertical asymptote



Example.

Determine the possible asymptotes of the following function:: $f(x)=\frac{x^2-4}{x+1}$

The first step in determining the presence of asymptotes is to determine the domain of the function:

$$x-1\neq 0\ \Rightarrow x\neq -1$$

Therefore, the domain is:

$$(-\infty,-1)\cup(-1,+\infty)$$

Since the function is not defined at a point, it makes sense to search for vertical and horizontal asymptotes:

$$\lim_{x\to -1^-}\frac{x^2-4}{x+1}=\frac{-3}{-1^-+1}=\frac{-3}{0^-}=+\infty$$

$$\lim_{x\to -1^+}\frac{x^2-4}{x+1}=\frac{-3}{-1^++1}=\frac{-3}{0^+}=-\infty$$

➡ $x=-1$ vertical as.

$$\lim_{x\to -\infty}\frac{x^2-4}{x+1}=\left[\frac{\infty}{\infty}\right]$$

$$\Rightarrow\lim_{x\to -\infty}\frac{x^2-4}{x+1}=\lim_{x\to -\infty}\frac{x^2}{x}=\lim_{x\to -\infty}x=-\infty$$

∄ left horiz. as.

$$\lim_{x\to +\infty}\frac{x^2-4}{x+1}=\left[\frac{\infty}{\infty}\right]$$

∄ right horiz. as.

$$\Rightarrow\lim_{x\to +\infty}\frac{x^2-4}{x+1}=\lim_{x\to +\infty}\frac{x^2}{x}=\lim_{x\to +\infty}x=+\infty$$

Example.

There is no horizontal asymptote → let us verify whether an oblique asymptote exists.

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{1}{x}\,\frac{x^2-4}{x+1} = \lim_{x\to-\infty} \frac{x^2-4}{x^2+x} = \left[\frac{\infty}{\infty}\right]$$

$$\Rightarrow m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{x^2}{x^2} = 1$$

$$q = \lim_{x\to-\infty} [f(x) - mx] = \lim_{x\to-\infty} \left[\frac{x^2-4}{x+1} - x\right] =$$

$$= \lim_{x\to-\infty} \left[\frac{\cancel{x^2} - 4 - \cancel{x^2} - x}{x+1}\right] = \lim_{x\to-\infty} \left[\frac{-4-x}{x+1}\right] = -1$$

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} = 1$$

$$q = \lim_{x\to-\infty} [f(x) - mx] = -1$$

→ $y = x - 1$

left oblique
as.

$$m = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2-4}{x^2+x} = 1$$

$$q = \lim_{x\to+\infty} [f(x) - mx] = \lim_{x\to+\infty} \left[\frac{-4-x}{x+1}\right] = -1$$

↓

$y = x - 1$

right oblique
as.

Discontinuities

- if f is defined on an interval $[a, b]$ excluding at most the point x_0

or

- if f is not continuous at a point x_0,

then x_0 is called a **singular point** or a **point of discontinuity**.

First-kind discontinuities (jump discontinuities)

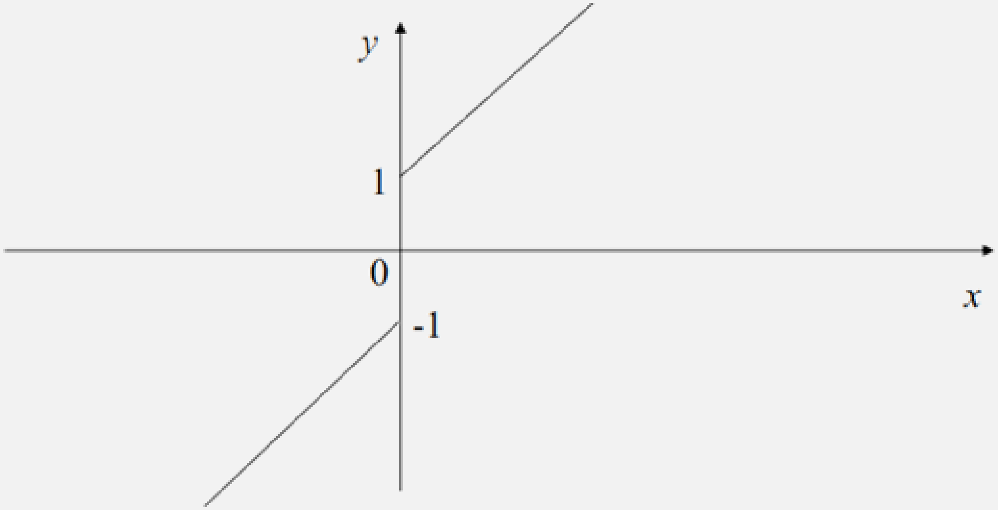
A function f has a first-kind discontinuity at x_0 if both the right-hand and left-hand limits exist and are finite, but are different:

$$l_1 = \lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x) = l_2$$

Example. The following function has a first-kind discontinuity at $x = 0$:

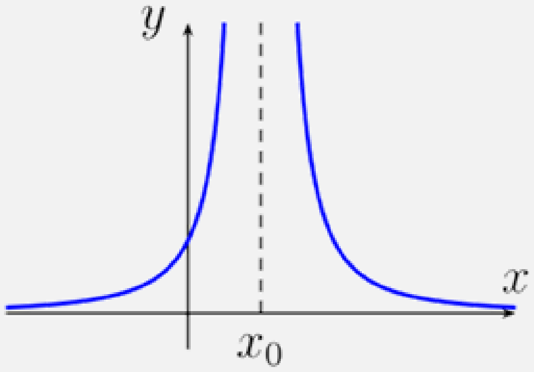
$$f(x) = x + \frac{|x|}{x} = \begin{cases} x + 1, se\ x > 0 \\ x - 1, se\ x < 0 \end{cases}$$

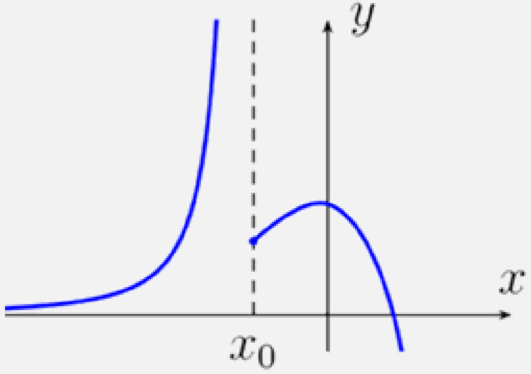
$$1 = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) = -1$$



Second-kind discontinuities

A function f has a second-kind discontinuity at x_0 if at least one of the one-sided limits is infinite or does not exist:

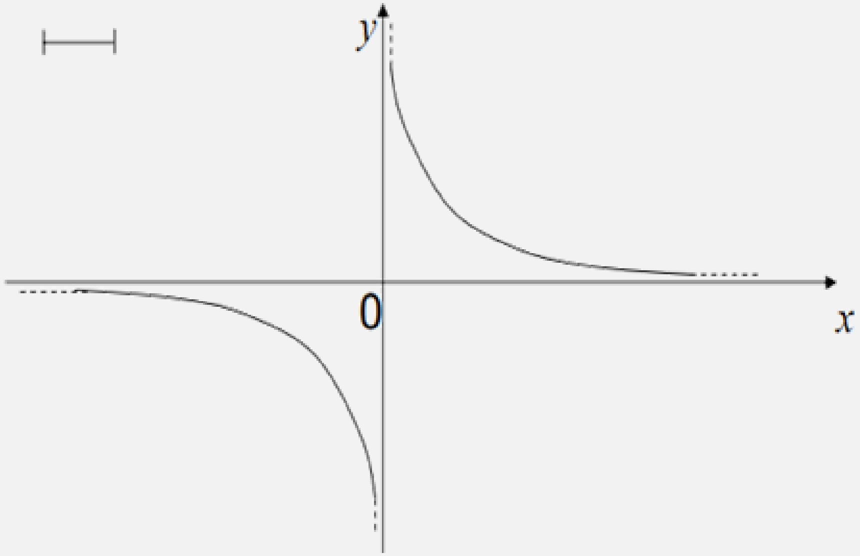




Example. The function has a second-kind discontinuity at $x = 0$:

$$f(x) = \frac{1}{x}$$

$$+\infty = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) = -\infty$$



The presence of vertical asymptotes is representative of second-kind discontinuities.

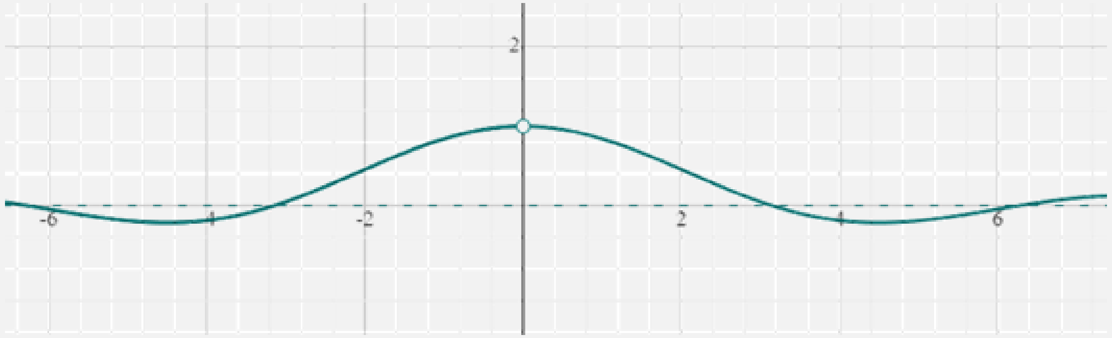
Third-kind (removable) discontinuities

A function f has a removable discontinuity at x_0 if the limit of f at x_0 exists and is finite, but:

- f is not defined at x_0
- $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$

Example. The function has a removable discontinuity at $x = 0$:

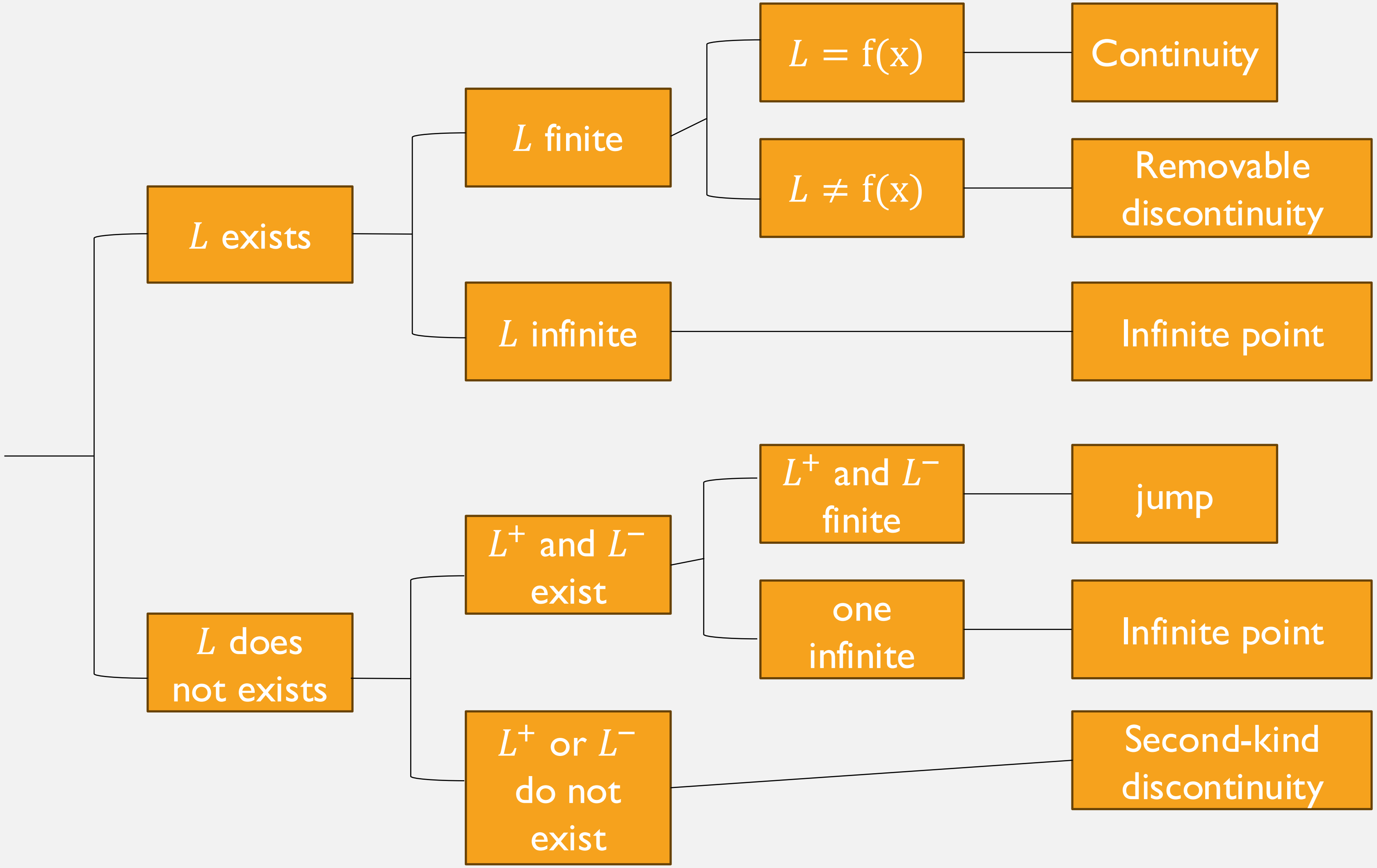
$$f(x) = \frac{\sin x}{x}$$



Indeed:

- *f is not defined at $x = 0 \Rightarrow \nexists f(0)$*
- *but there exists* $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Punti di discontinuità



L exists

L finite

$L = \mathrm{f}(\mathrm{x})$ — Continuity

$L \neq \mathrm{f}(\mathrm{x})$ — Removable discontinuity

L infinite — Infinite point

L does not exists

L^+ and L^- exist

L^+ and L^- finite — jump

one infinite — Infinite point

L^+ or L^- do not exist — Second-kind discontinuity

Remarkable limits

Limit	Indeterminate form
$\lim_{x\to 0}\frac{\sin x}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$	$\frac{0}{0}$
$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$	1^∞
$\lim_{x\to 0^+} x\cdot \ln x=0$	$0\cdot\infty$
$\lim_{x\to 0}\frac{e^x-1}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\arcsin x}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\arctan x}{x}=1$	$\frac{0}{0}$

Limit	Indeterminate form
$\lim_{x\to 0}\frac{\tan x}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{1-\cos x}{x}=0$	$\frac{0}{0}$
$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$	1^∞
$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{(1+x)^k-1}{x}=k$	$\frac{0}{0}$