- **DERIVATIVE OF A FUNCTION**

# Definitions: interior point and difference quotient

Let $I$ be a nonempty interval. We say that $x_0 \in I$ is an **interior point** of $I$ if $\exists r > 0 : (x_0 - r, x_0 + r) \subset I$.
Moreover, $I \subseteq D_f$ (the function is defined at every point of $I$).

Let $f : I \to \mathbb{R}$ and $x_0 \in I$ be an interior point of $I$. Given $h \in \mathbb{R}, h \neq 0$, we call the **difference quotient** of $f$ relative to $x_0$ and to the increment $h$ the ratio:

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

negative difference quotient → $f(x)$ decreases going from $x$ to $x + h$

positive difference quotient → $f(x)$ increases going from $x$ to $x + h$

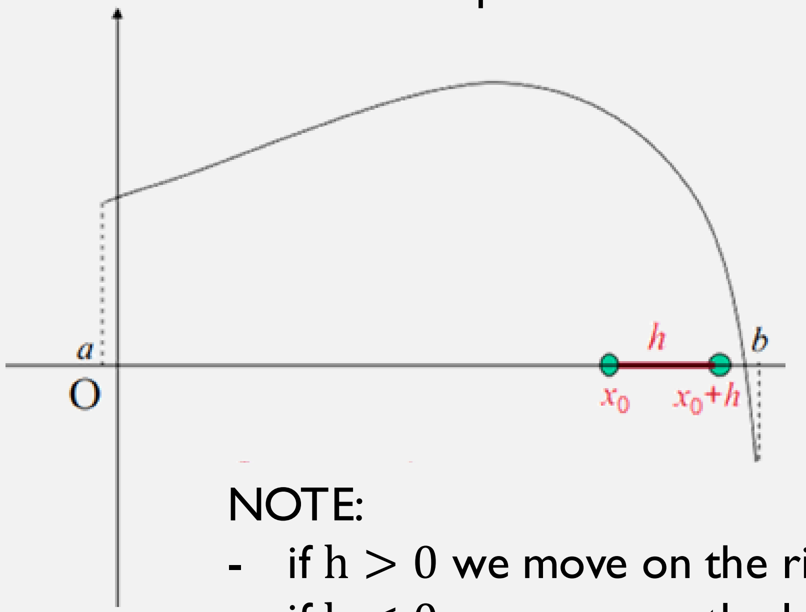
# Increment of the variable $x$

Be $f(x)$ a function in an interval $[a, b]$ and be $x_0$ a fixed interior point of the interval $[a, b]$.

We pass from the point $x_0$ to another interior point of the interval $[a, b]$.

⇩

we provide a certain increment, called $h$, to the value $x_0$ , in a way that we still are within the interval $[a, b]$




NOTE:

- if $h > 0$ we move on the right of $x_0$
- if $h < 0$ we move on the left of $x_0$

The passage from $x_0$ to $x_0 + h$ along the $x$-axis is called **increment of the variable $x$** and corresponds to the value $\boldsymbol{h = (x_0 + h) - x_0}$

# Increment of the function $f(x)$

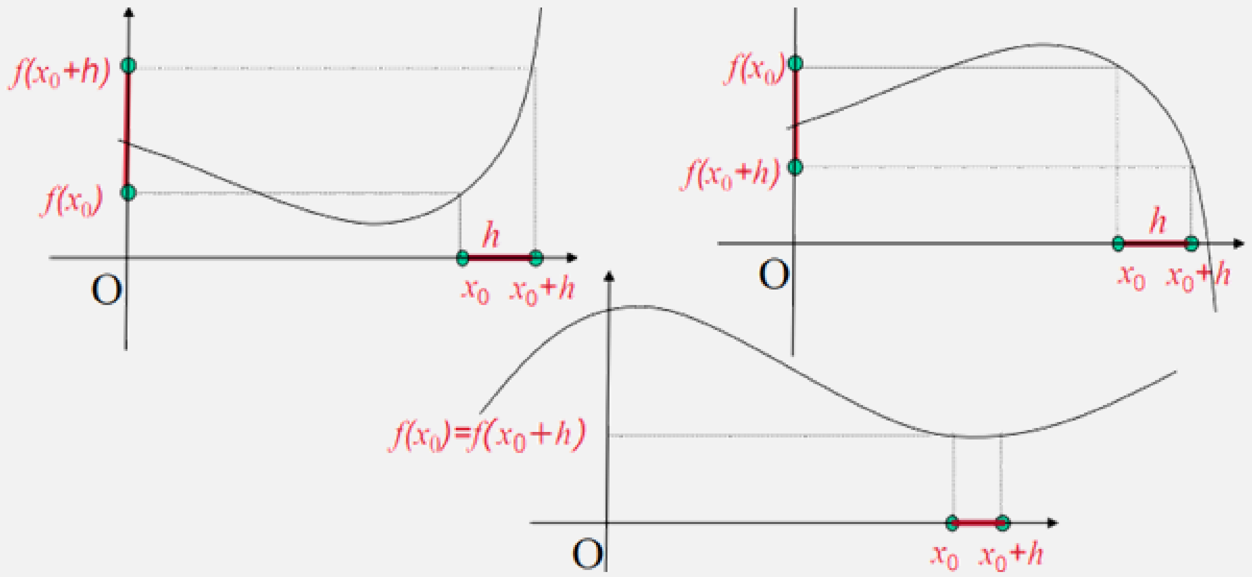
The images through $f$ of the points $x_0$ and $x_0 + \mathrm{h}$ are respectively $f(x_0)$ and $f(x_0 + h)$

The difference $f(x_0 + h) - f(x_0)$ between the values assumed by the function in passing from $x_0$ to $x_0 + \mathrm{h}$ is called

**increment of the function $f$**

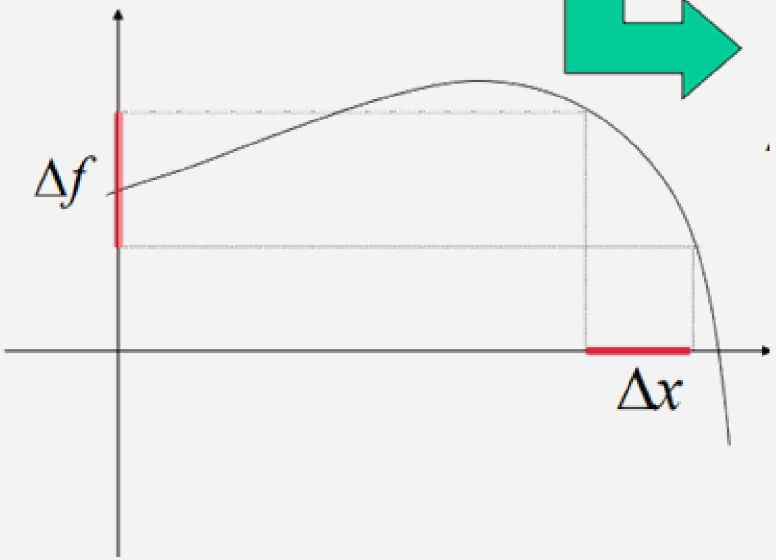
We calculate the value of the increment of the function $f$ in passing from $x_0$ to $x_0 + \mathrm{h}$ with $x_0 < x_0 + \mathrm{h}$

- if $f(x_0) < f(x_0 + h)$, positive increment, $f$ increases passing from $x_0$ to $x_0 + \mathrm{h}$
- if $f(x_0) > f(x_0 + h)$, negative increment, $f$ decreases passing from $x_0$ to $x_0 + \mathrm{h}$
- if $f(x_0) = f(x_0 + h)$, null increment, $f$ is constant passing from $x_0$ to $x_0 + \mathrm{h}$



# Difference quotient

the increment of the variable $x$ is indicated by the symbol $\Delta x$
the increment of the function $f$ is indicated by the symbol $\Delta f$



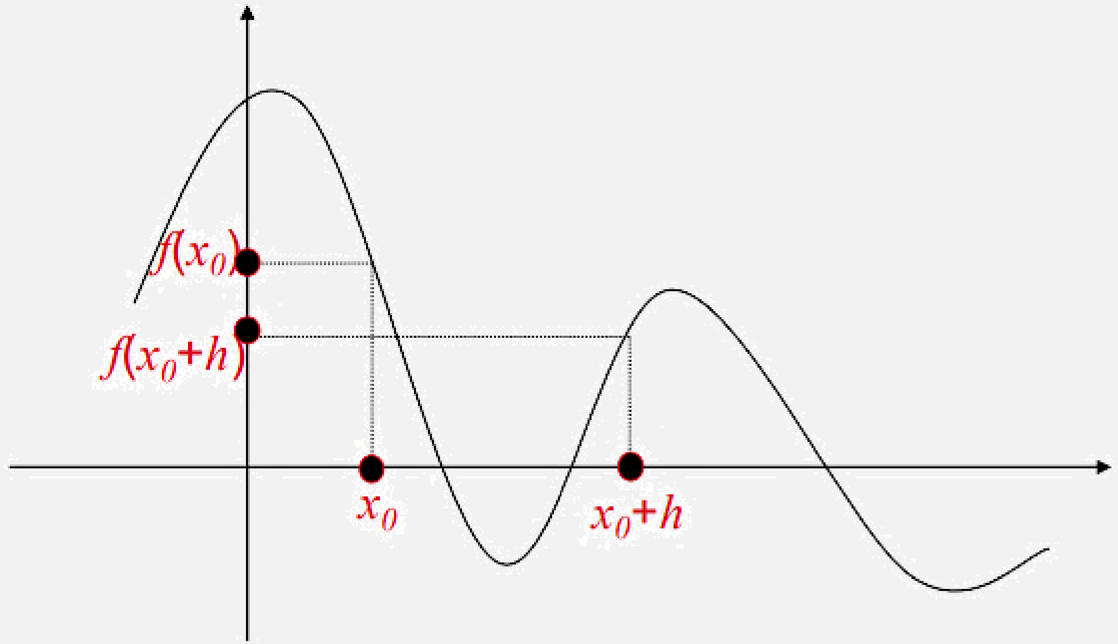

$$\Delta x = x_0 + h - x_0$$
$$\Delta f = f(x_0 + h) - f(x_0)$$

the ratio between the increment of the variable $x$ passing from $x_0$ to $x_0 + \mathrm{h}$ and the increment of the function $f$ is called **difference quotient** of the function relating to the passage from $x_0$ to $x_0 + \mathrm{h}$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

*the difference quotient expresses the "variability of $f$" respect to a certain interval*

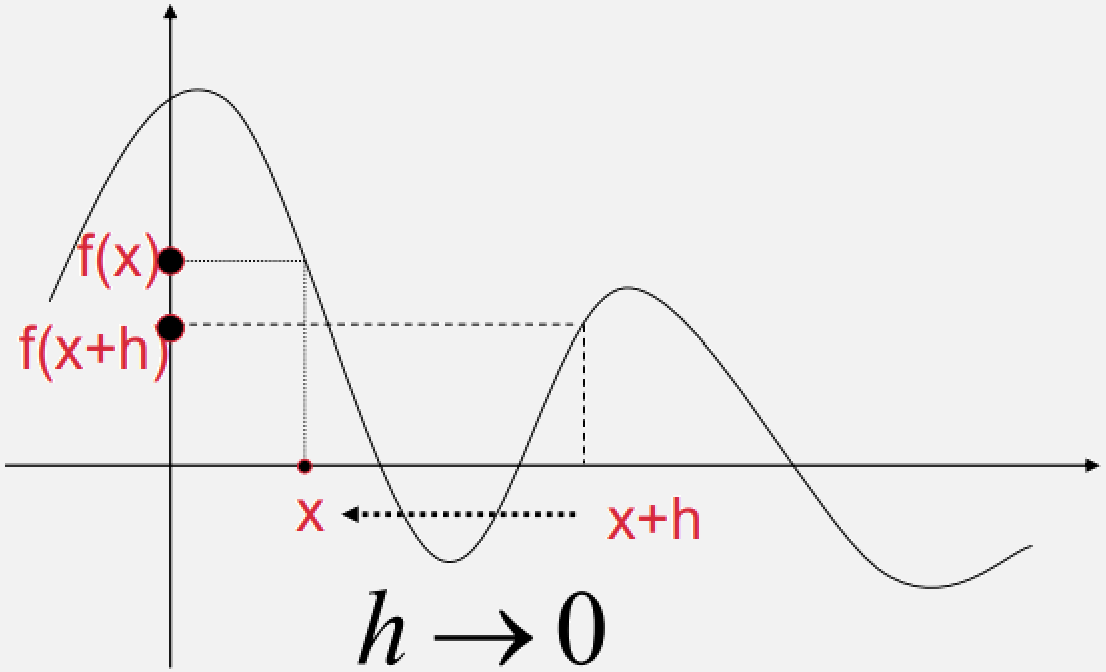
# Difference quotient




From the graph it is evident that the difference quotient for the passage from $x_0$ to $x_0 + h$ is negative.
Therefore, the function decreases from $x_0$ to $x_0 + h$, but it does not necessarily decrease over the entire interval $[x_0 , x_0 + h]$.

Hence, the difference quotient provides information about monotonicity only for the specific interval, not about the behavior around a single point.

# Difference quotient → pointwise information




Pointwise information about monotonicity at $x_0$ requires considering a very small interval $[x_0\,, x_0 + h]$, which is obtained when the point $x_0 + h$ gets «closer» to $x_0$ , that is to say as $h \to 0$.

## Definition: derivative

Let $f(x)$ be defined on $[a, b]$ and let $x_0$ be an interior point.

The derivative of $f(x)$ at $x_0$ is defined as the limit, if it exists and is finite, of the difference quotient of $f(x)$ passing from $x$ to $x + h$, that is to say as h $h \to 0$ :

$$D\big(f(x_0)\big) = \frac{df(x_0)}{dx} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Derivative of the function $f$ in $x$

## Definition: derivative

$$D(f(x_0)) = \frac{df(x_0)}{dx} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative of a function $f(x)$ at an interior point $x_0$ is a real number, when it exists.

If the function $f(x)$ is defined in an interval $[a, b]$, at the endpoints of the interval only the right derivative (at $a$) and the left derivative (at $b$) can be defined.

A function is **differentiable** on an interval $[a, b]$ if it is differentiable at every interior point of $[a, b]$ and if admits a right derivative at $a$ and a left derivative at $b$.

## Differentiability and continuity

What is the link between differentiability and continuity of a function in $x_0$?

Theorem.

If a function $f(x)$ is differentiable in a point $x_0 \in (a, b)$, then $f(x)$ is continuous in $x_0$

Hence, the differentiability in a certain point implies the continuity in that point.
However, **the converse is NOT always true**: a function continuous in a point $x_0$ can also be not differentiable in $x_0$

**Example**.

$f(x) = |x - 2|$ is defined and continuous in $\mathbb{R}$,
hence also in $x_0 = 2$

Let us verify the differentiability:

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{f(2+h)-f(2)}{h} =$$

$$= \frac{|2+h-2|-|2-2|}{h} = \frac{|h|}{h}$$ passando al limite per h->0

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

$$\lim_{h \to 0} \frac{|h|}{h} =$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

since the function admits in $x_0 = 2$ right and left derivatives finite but different from each other, the function is not differentiable in this point (even if it is continuous in this point)

## Differentiability and continuity

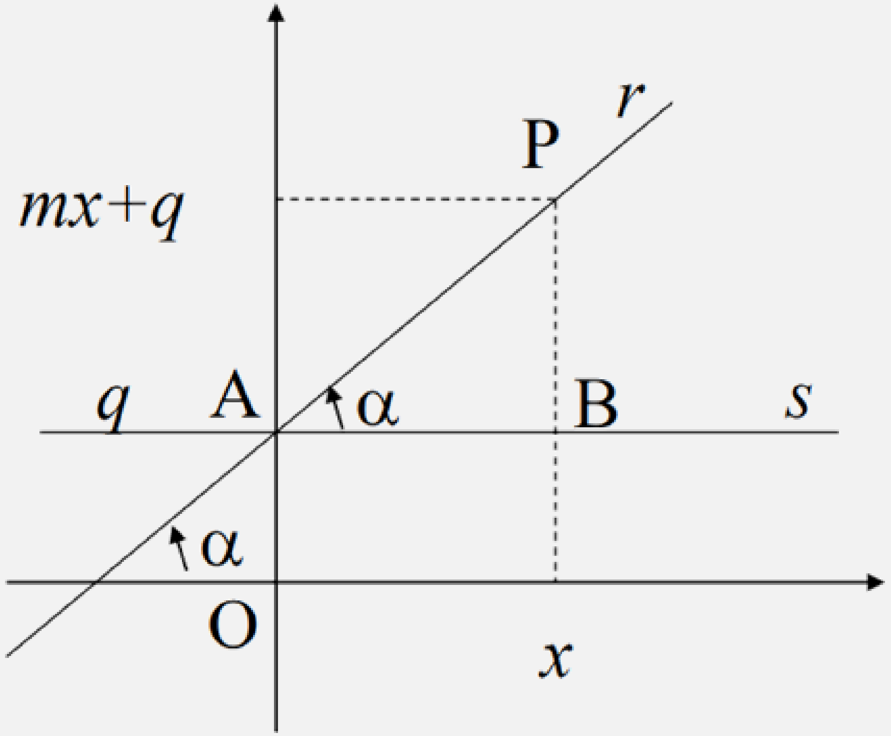
If a function $f(x)$ is differentiable in a point $x_0 \in (a, b)$, then $f(x)$ is continuous in $x_0$

from this theorem, it immediately follows that: in the points of discontinuity, a function does not admit derivative (it is not differentiable)

therefore, if $f(x)$ is not continuous in $x_0 \in (a, b)$, then $f(x)$ is not differentiable in $x_0$

# Geometric meaning of the derivative

Remember:
Let $r$ be a line non-parallel to the $y$-axis, described by the equation $y = mx + q$, with $m$ being the angular coefficient and $q$ the constant term




The line $r$: $y = mx + q$ forms the angle $\alpha$ with the $x$-axis, and the point $A$ has coordinates $A = (0, q)$.

The generic point $P$ on the line $r$ has coordinates $P = (x, mx + q)$.

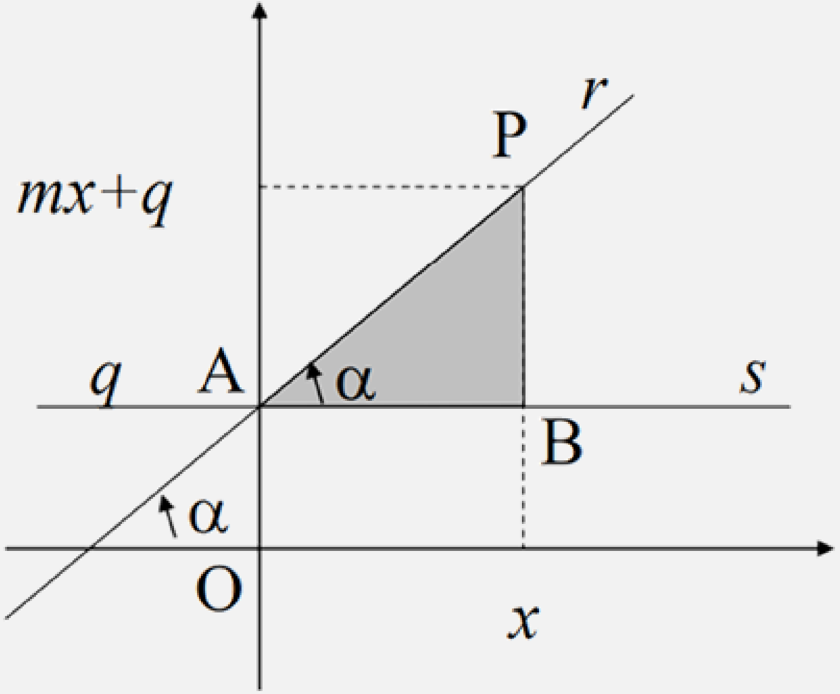
Let $s$ be the line passing through the point $A$ and parallel to the $y$-axis.

Finally, let $B$ be the point of intersection between the line $s$ and the perpendicular segment from $P$ to the $y$-axis, having coordinates $B(x, q)$.

The angle $PAB = \alpha$

# Geometric meaning of the derivative

Remember:
Geometric meaning of the angular coefficient of a line




$$A = (0, q)$$
$$P(x, mx + q)$$
$$B(x, q)$$
$$PAB = \alpha$$

Let us consider the right-angle triangle in $B$:

$$BP = AP \text{ sen } \alpha \qquad AB = AP \cos \alpha$$

$$\frac{BP}{AB} = \frac{AP \text{ sen } \alpha}{AP \cos \alpha} = \frac{\text{sen } \alpha}{\cos \alpha} = \tan \alpha$$

$$\Rightarrow \frac{BP}{AB} = \tan \alpha$$

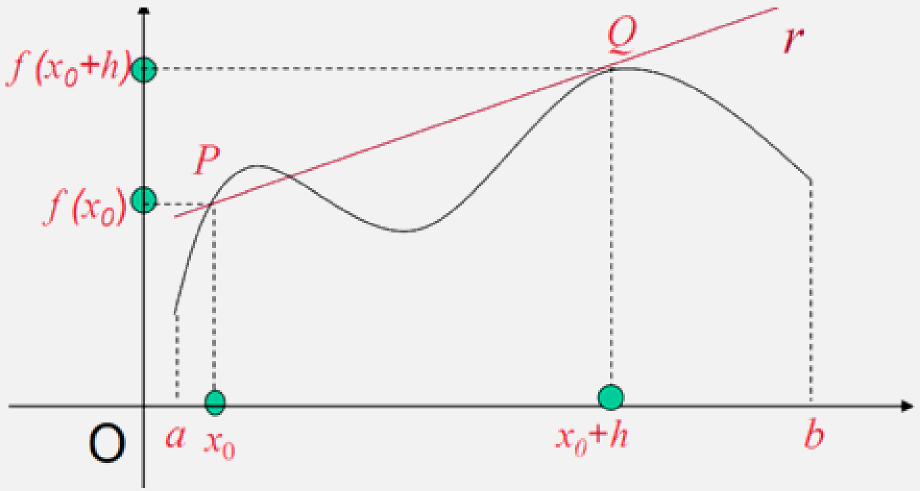
$$BP = (mx + q) - q = mx \qquad AB = (x)-0=x$$

$$\frac{BP}{AB} = \frac{mx}{x} = m$$

$$\Rightarrow \frac{BP}{AB} = m$$

# Geometric meaning of the derivative

Let $f(x)$ be a function differentiable on an interval $[a,b]$ and let $x_0$ be a fixed interior point of the interval $[a,b]$. We move from the point $x_0$ to another point $x_0+h$ inside the interval $[a,b]$ so that the corresponding values of the function $f$ can be considered.




we indicate the two points belonging to the graph of $f$ as:

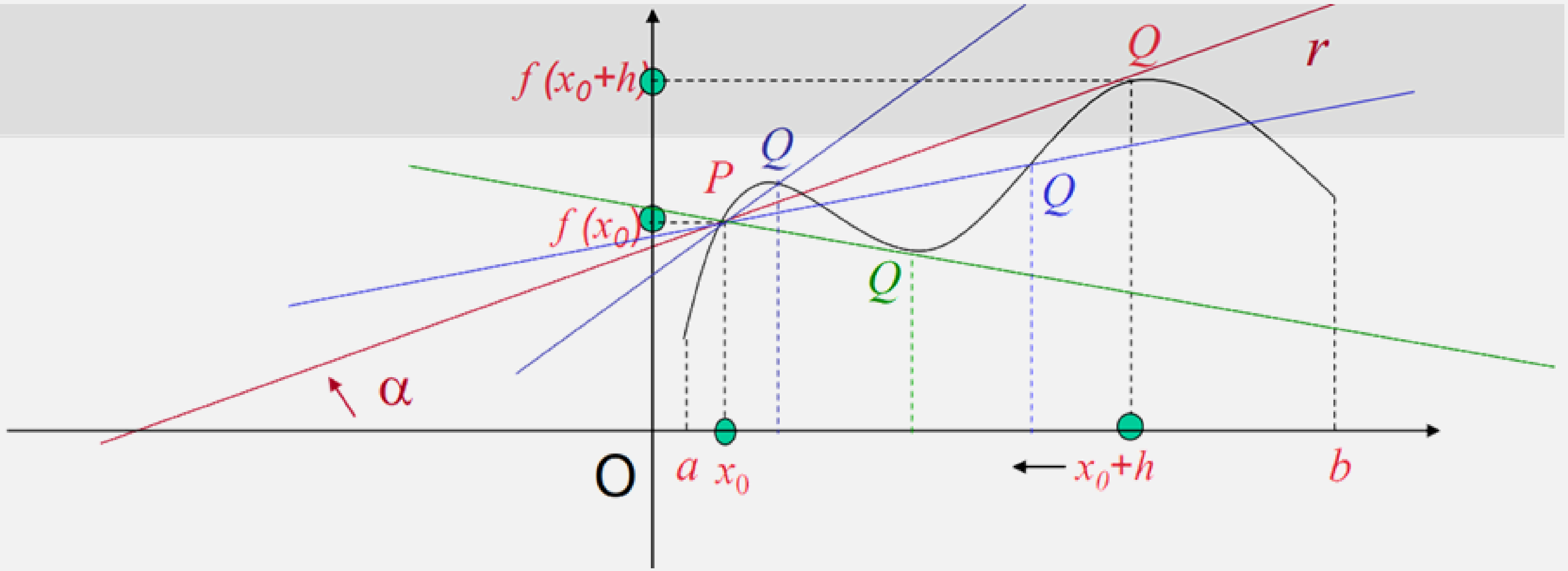
$$P(x_0, f(x_0))$$
$$Q(x_0+h, f(x_0+h))$$

Let $r$ be the line passing through $P$ and $Q$ and forming an angle $\alpha$ with the positive semi-axis of $x$ values

## Geometric meaning of the derivative

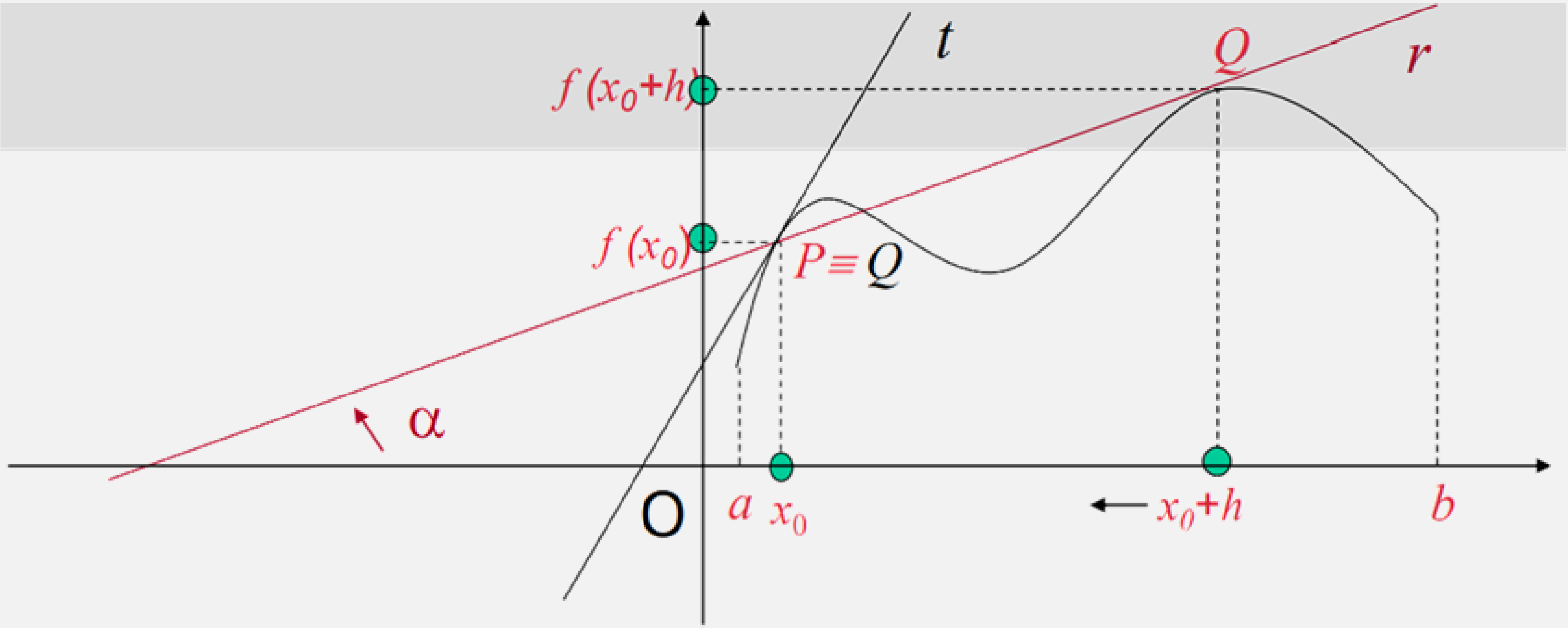
From a geometric point of view, as $h \to 0$ the point $P(x_0, f(x_0))$ remains fixed, while the point $Q(x_0 + h, f(x_0 + h))$ moves toward $P$ along the graph of the function $f$.

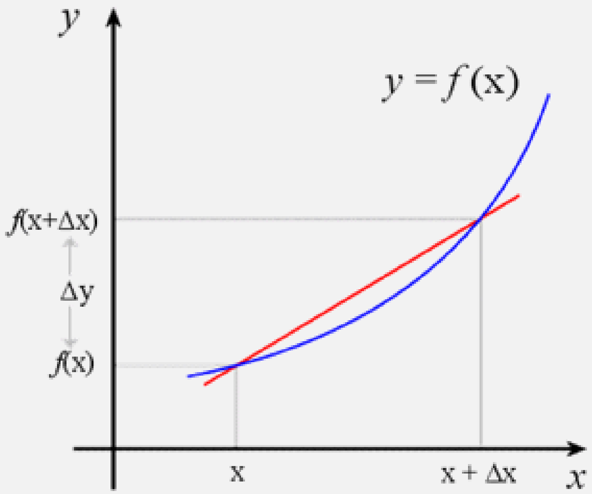
At the same time, as point $Q$ approaches point $P$, the line passing through $P$ and $Q$ changes, in particular **its slope changes.**



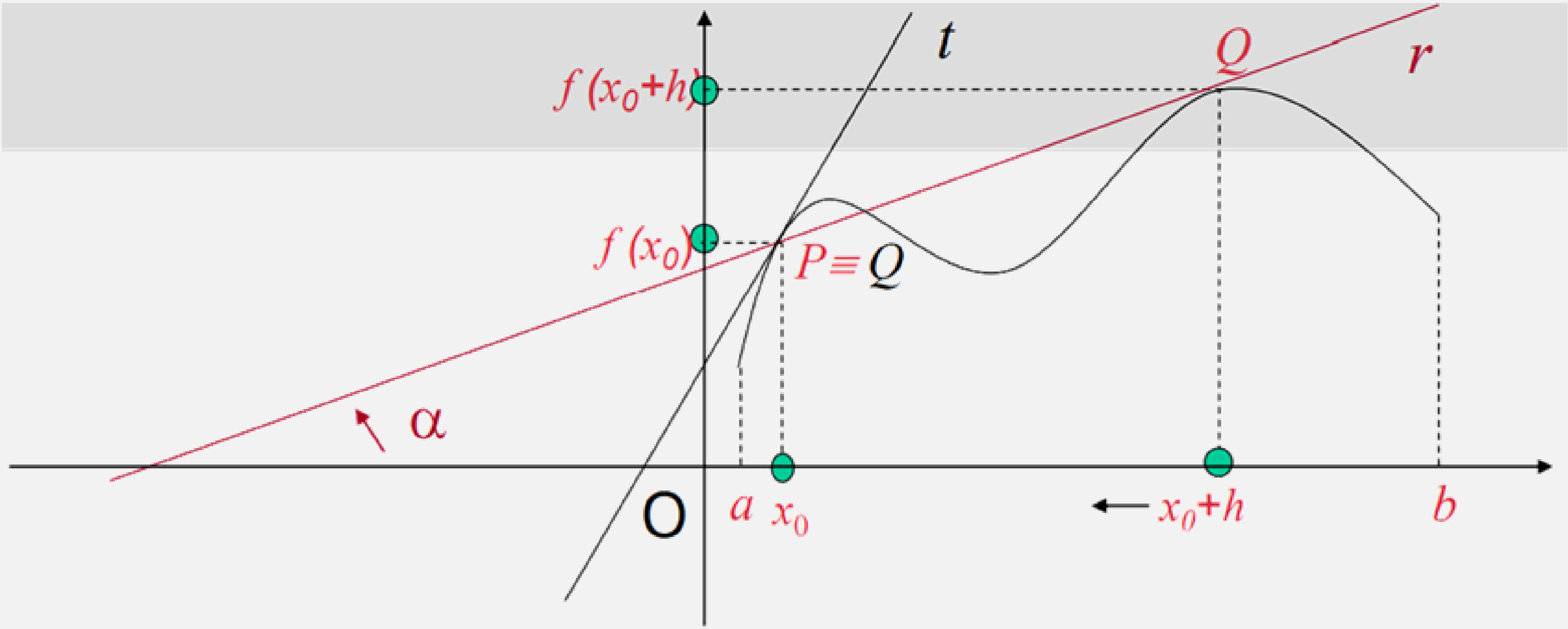
# Geometric meaning of the derivative

These variations, as $h \to 0$, cease when point $Q$ coincides with point $P$.
At that moment, the line passing through $P$ and $Q$ approaches a limiting position, which is identified as the tangent line to the graph of the function $f$ at the point with abscissa $x_0$ that is, at point $P$.



# Geometric meaning of the derivative




Let α be the angle formed by the tangent line with the positive $x$-axis, and let $m_t$ be its slope.

Then:

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h \to 0} \tan\alpha = \lim_{h \to 0} m_r \Leftrightarrow f'(x_0) = \tan\alpha = m_t$$

# Geometric meaning of the derivative

Therefore, if $\alpha$ is the angle formed by the tangent line to the graph of $f$ at the point with abscissa $x_0$, and $m_t$ is its slope, then:

$$f'(x_0) = \tan\alpha = m_t$$

That is:

The derivative $f'(x_0)$ of the function $f$ in the point $x_0$ is equal to the angular coefficient $m_t$ of the tangent line to the graph at the point $P\big(x_0, f(x_0)\big)$

**Conclusions**.

The existence of the derivative of a function $f$ at a point $x_0$ is related to:

- the existence of the line tangent to the graph of $f$ in the point with abscissa $x_0$
- the fact that the slope of the tangent line must be finite, since $f'(x_0) = m_t$

In particular, since $f'(x_0) = \tan\alpha = m_t$, requiring the slope of the tangent line $t$ to be finite is equivalent to requiring:

$$\alpha \neq \frac{\pi}{2}$$

If $\alpha \to \frac{\pi}{2} \Longrightarrow \tan\alpha \to \pm\infty$

The tangent line to the graph of the function $f$ at a point with abscissa $x_0$ cannot be parallel to the $y$-axis if the function is to be differentiable at $x_0$

If $f$ is differentiable in a point $x_0$ , then in the point of coordinates $(x_0 , f(x_0))$ its graph admits a tangent line non-parallel to the $x$-axis

A function differentiable on an interval is a function whose graph admits a tangent line at every point

- **DERIVATIVES OF ELEMENTARY FUNCTIONS**

## Derivatives of elementary functions

If a function $f$ is differentiable at every point of the interval $(a,b)$, it is possible to define a new function that associates to each $x \in (a,b)$ the value of the derivative $f'(x)$.

$$x \in (a,b) \to f'(x) \in \mathbb{R}$$

This function is called the **derivative function** and is denoted by $\boldsymbol{f'(x)}$

# Derivative of a constant function

Let $f(x) = k$, with $k \in \mathbb{R}$.

What is its derivative $\forall x \in \mathbb{R}$?

$$\frac{f(x+h) - f(x)}{h} = \frac{k - k}{h} = \frac{0}{h} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 0 = 0$$

$$\Rightarrow Dk = 0, \forall x \in \mathbb{R}$$

## Geometric interpretation

$$Dk = 0, \forall x \in \mathbb{R}$$

This result has a geometric interpretation: the graph of the constant function $f(x) = k$ is a line parallel to the $x$-axis

↓

At every point, the tangent line coincides with the graph and has slope:

↓

$$m_t = 0, \forall x \in \mathbb{R}$$

(in fact, $\tan 0 = 0$)

# Derivative of a linear function

Let $f(x) = x$ be the bisector line of I and III quadrants.

What is its derivative $\forall x \in \mathbb{R}$?

$$\frac{f(x+h) - f(x)}{h} = \frac{x + h - x}{h} = \frac{h}{h} = 1$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 1 = 1$$

$$\Rightarrow \quad Dx = 1, \forall x \in \mathbb{R}$$

## Geometric interpretation

$$Dx = 1, \forall x \in \mathbb{R}$$

This result has a geometric interpretation: the graph of the function $f(x) = x$ is the bisector line of I and III quadrants

↓

in every point, the tangent line coincides with the graph of the function

↓

the angular coefficient of the tangent line is

↓

$$m_t = 1, \forall x \in \mathbb{R}$$

(in fact, $\tan\frac{\pi}{4} = 1$)

# Derivative of a power function

Let the function $f(x) = x^{\alpha}, \alpha \in \mathbb{R}, x > 0$.

It can be shown that the derivative of this function, for all $x > 0$ ($\forall x \in \mathbb{R}^{+}$) and for all $\alpha \in \mathbb{R}$, is:

$$Dx^{\alpha} = \alpha x^{\alpha-1}, \forall x > 0, \forall \alpha \in \mathbb{R}$$

Example.
$f(x) = x^2$ in $x_0 = 2$

$$\frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{(2+h)^2-(2)^2}{h} = \frac{4+4h+h^2-4}{h} = \frac{h^2+4h}{h} = h+4$$

$$\Rightarrow \lim_{h\to 0} h+4 = 4$$

The function $f(x) = x^2$ is differentiable in $x_0 = 2$ and $f'(2) = 4$

# Derivative of a power function

Let the function $f(x) = \sqrt{x} = x^{\frac{1}{2}}$.

Recalling the rule: $Dx^{\alpha} = \alpha x^{\alpha-1}$

$$f(x) = x^{\frac{1}{2}} \rightarrow f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

$$D\sqrt{x} = \frac{1}{2\sqrt{x}}, \forall x \in \mathbb{R}^{+}$$

# Derivative of a power function

Let the function $f(x) = \frac{1}{x} = x^{-1}$.

Recalling the rule: $Dx^{\alpha} = \alpha x^{\alpha - 1}$

$$f(x) = x^{-1} \rightarrow f'(x) = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$D\frac{1}{x} = -\frac{1}{x^2}, \forall x \in \mathbb{R}^+$$

## Derivatives of elementary functions: differentiation rules

$$D a^x = a^x \log a$$

$$D e^x = e^x$$

$$D \log_a x = \frac{1}{x} \cdot \frac{1}{\log a}$$

$$D \log x = \frac{1}{x}$$

$$D \sin x = \cos x$$

$$D \cos x = -\sin x$$

# Derivatives of elementary functions

| $\boldsymbol{f(x)}$ | $\boldsymbol{f'(x)}$ |
|---|---|
| $k$ | $0$ |
| $x$ | $1$ |
| $x^n$ | $nx^{n-1}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ | $a_1 + 2a_2x + \cdots + na_nx^{n-1}$ |
| $\log_a x$ | $\frac{1}{x} \cdot \log_a e = \frac{1}{x} \cdot \frac{1}{\ln a}$ |
| $\log_e x$ | $\frac{1}{x} \cdot \log_e e = \frac{1}{x}$ |
| $a^x$ | $a^x \log_e a = a^x \cdot \frac{1}{\log_a e}$ |
| $e^x$ | $e^x \log_e e = e^x$ |

| $\boldsymbol{f(x)}$ | $\boldsymbol{f'(x)}$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\frac{1}{\cos^2 x} = 1 + \tan^2 x$ |
| $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan x$ | $\frac{1}{1+x^2}$ |
| $\lvert x\rvert$ | $\frac{x}{\lvert x\rvert} = \frac{\lvert x\rvert}{x}$ |
| $\lvert f(x)\rvert$ | $f'(x)$ con $f(x) > 0$<br>$-f'(x)$ con $f(x) < 0$ |

**Exercises**. Compute the derivative of the following functions.

1. $f(x) = x^7$ $\quad f'(x) = 7x^{7-1} = 7x^6$
2. $f(x) = x^{-2}$ $\quad f'(x) = -2x^{-2-1} = -2x^{-3}$
3. $f(x) = x^{\pi}$ $\quad f'(x) = \pi x^{\pi-1}$
4. $f(x) = 4$ $\quad f'(x) = 0$
5. $f(x) = \sqrt[3]{x^2}$ $\quad f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$
6. $f(x) = \frac{1}{\sqrt{x}}$ $\quad f'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$
7. $f(x) = x^{-2}$ $\quad f'(x) = -2x^{-2-1} = -2x^{-3}$
8. $f(x) = 3^x$ $\quad f'(x) = 3^x \log_e 3$
9. $f(x) = \log_5|x|$ $\quad \frac{1}{|x|} \cdot \log_5 e = \frac{1}{x} \cdot \frac{1}{\ln 5}$
10. $f(x) = e$ $\quad f'(x) = 0$

## Second derivative – n-th derivative

At this point, it is natural to ask whether the derivative function $f'(x)$ is itself differentiable at a point or on the entire interval $(a, b)$.
If this is the case, we call the derivative of $f'$ the **second derivative**, denoted by:

$$f''(x)$$

$$D^2 f(x)$$

Similarly, we define the third derivative $f'''(x)$, fourth derivative $f^{IV}(x)$, and in general the n$^{th}$ derivative:

$$f^n(x)$$

$$D^n f(x)$$

- **RULES OF DIFFERENTIATION**

# Differentiation rules (sum, product, quotient)

Let $f : (a, b) \to \mathbb{R}, g : (a, b) \to \mathbb{R}$ be two functions differentiable on $(a, b)$. Then the functions:

$$f \pm g, \quad f \cdot g, \quad \frac{f}{g}, \text{(with } g \neq 0)$$

are differentiable in $(a, b)$, and the following formulas hold:

- $(f \pm g)' = f' \pm g'$
- $(f \cdot g)' = f' \cdot g + f \cdot g' \to (k \cdot f)' = k \cdot f'$
- $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \to \left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

**Exercises**. Compute the derivative of the following functions

$f(x) = 4x^2 + 4$

$\frac{d}{dx}(4x^2) + \frac{d}{dx}(4) =$

$f'(x) = 4 \cdot 2x^{2-1} + 0 = 8x$

$f(x) = \sqrt{x} + x^{2/3}$

$\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}\left(x^{\frac{2}{3}}\right)$

$f'(x) = \frac{1}{2}x^{-1/2} + \frac{2}{3}x^{-1/3} = \frac{1}{2x^{1/2}} + \frac{2}{3x^{1/3}}$

$f(x) = -x^{-2} + 3x^3$

$\frac{d}{dx}(-x^{-2}) + \frac{d}{dx}(3x^3) =$

$f'(x) =- -2x^{-2-1} + 3 \cdot 3x^{3-2} = 2x^{-3} + 9x^2$

**Exercises**. Compute the derivative of the following functions

$$f(x) = \log x - e^x$$

$$\frac{d}{dx}(\ln x) - \frac{d}{dx}(e^x)$$

$$f'(x) = \frac{1}{x} - e^x$$

$$f(x) = x \cdot e^x$$

$$e^x \frac{dx}{dx} + x \frac{d}{dx}(e^x)$$

$$f'(x) = 1 \cdot e^x + x \cdot e^x = e^x + xe^x$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$f(x) = x^2 \cdot (x+1)$$

$$(x+1)\frac{d}{dx}x^2 + x^2 \frac{d}{dx}(x+1)$$

$$f'(x) = 2x \cdot (x+1) + x^2 \cdot (1) = 2x^2 + 2x + x^2 = 3x^2 + 2x$$

$$f(x) = \frac{x-1}{x}$$

$$\frac{\frac{d}{dx}(x-1) \cdot x - \frac{dx}{dx}(x-1)}{x^2}$$

$$f'(x) = \frac{1 \cdot x - 1 \cdot (x-1)}{x^2} = \frac{1}{x^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \rightarrow \left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

## Theorem: derivative of composite functions (Chain Rule)

Let the composite function $f\big(g(x)\big)$ be defined by two functions $f(x)$ and $g(x)$.

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composite function $f\big(g(x)\big)$ is differentiable and the followinf formula holds:

$$(f \circ g)' = f'\big(g(x)\big) \cdot g'(x)$$

$$(f \circ g)' = f'\big(g(x)\big) \cdot g'(x)$$

**Esempi.**

$$f(x) = (3x+1)^2 \;\rightarrow f'(x) = 2 \cdot (3x+1)^{2-1} \cdot 3 = 6(3x+1)$$

$$f(x) = (1+2x^2)^3 \;\rightarrow f'(x) = 3 \cdot (1+2x^2)^{3-1} \cdot [2 \cdot 2x^{2-1}] = 12x(1+2x^2)^2$$

$$f(x) = \sqrt{(4x+3)} = (4x+3)^{\frac{1}{2}} \;\rightarrow f'(x) = \frac{1}{2} \cdot (4x+2)^{\frac{1}{2}-1} \cdot 4 = 2 \cdot (4x+2)^{-\frac{1}{2}} = \frac{2}{\sqrt{4x+3}}$$

$$f(x) = \log x^2 \;\rightarrow f'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$

# Differentiation rules for composite functions

| $\boldsymbol{f(x)}$ | $\boldsymbol{f'(x)}$ |
|---|---|
| $[f(x)]^{\alpha}$ | $\alpha \cdot f(x)^{\alpha-1} \cdot f'(x)$ |
| $\sqrt{f(x)}$ | $\frac{1}{2\sqrt{f(x)}} \cdot f'(x)$ |
| $\frac{1}{f(x)}$ | $-\frac{1}{f^2(x)} \cdot f'(x)$ |
| $\alpha^{f(x)}$ | $\alpha^{f(x)} \log \alpha \cdot f'(x)$ |
| $e^{f(x)}$ | $e^{f(x)} \cdot f'(x)$ |
| $\log_a f(x)$ | $\frac{1}{f(x)} \cdot \frac{1}{\log a} \cdot f'(x)$ |
| $\log f(x)$ | $\frac{1}{f(x)} \cdot f'(x)$ |

## De l'Hôpital's Theorem

Let $f$ and $g$ be two functions differentiable on $(a, b)$ such that: $g(x), g'(x) \neq 0, \forall x \in (a, b)$, except possibly at the point $x_0 \in (a, b)$.
If it holds that:

1. $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$

2. $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$ with $l$ finite or infinite

Then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$$

## De l'Hôpital's Theorem

Observation 1.
The De l'Hôpital's theorem still holds even if the interval $(a, b)$ where $f$ and $g$ are differentiable is not bounded and if $x_0 = \pm\infty$

Observation 2.
The De l'Hôpital's theorem can be applied more than once consecutively if, after applying it the first time one finds another indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

# De l'Hôpital's Theorem

Examples.

$$\lim_{x\to 1}\frac{x^2+x-2}{x-1}=\frac{0}{0}=\lim_{x\to 1}\frac{2x+1}{1}=3$$

$$\lim_{x\to 1}\frac{\log x}{x-1}=\frac{0}{0}=\lim_{x\to 1}\frac{1/x}{1}=1$$

$$\lim_{x\to +\infty}\frac{\log(1+x^5)}{\log(2+x^3)}=\frac{\infty}{\infty}=\lim_{x\to +\infty}\frac{\frac{5x^4}{1+x^5}}{\frac{3x^2}{2+x^3}}=\lim_{x\to +\infty}\frac{5x^4}{1+x^5}\frac{2+x^3}{3x^2}=\frac{10x^4+5x^7}{3x^2+3x^7}=\frac{5}{3}$$

## De l'Hôpital's Theorem

Examples.

$$\lim_{x\to 1^+} [\log x \cdot \log(x-1)] = 0 \cdot \infty$$

*You cannot apply the theorem!*

$$\lim_{x\to 1^+} \frac{\log(x-1)}{1/\log x} = \frac{\infty}{\infty}$$

*You can apply the theorem!*

$$\lim_{x\to 1^+} \frac{\log(x-1)}{1/\log x} = \frac{\infty}{\infty} = \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{-\frac{1}{\log^2 x}\cdot\frac{1}{x}} = \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{-\frac{1}{x\log^2 x}} = \lim_{x\to 1^+} \frac{-x\log^2 x}{x-1} = \frac{0}{0}$$

$$\lim_{x\to 1^+} \frac{-x\log^2 x}{x-1} = \frac{0}{0} = \lim_{x\to 1^+} \frac{-\left(\log^2 x + x\cdot 2\log x\cdot\frac{1}{x}\right)}{1} = 0$$

- **NON-DIFFERENTIABLE FUNCTIONS**
- **Corner points**
- **Inflection points with vertical tangent**
- **Cusps**

# Function not differentiable at a point: corner point

$$f(x) = |x|$$

This function is defined and continuous on all of $\mathbb{R}$; in particular, it is defined and continuous at $x_0 = 0$.

Let us verify whether $f(x)$ is also differentiable at $x_0 = 0$.

Compute the difference quotient with $x_0 = 0$:

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{|h| - |0|}{h} = \frac{|h|}{h}$$

$$\lim_{h \to 0} \frac{|h|}{h} = \begin{array}{l} \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1 \\ \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1 \end{array}$$

## Function not differentiable at a point: corner point

The function $f(x) = |x|$ admits in the point $x_0 = 0$ right and left derivatives both finite but distinct:

$$f'_+(x_0) = 1 \text{ and } f'_-(x_0) = -1$$

⇓

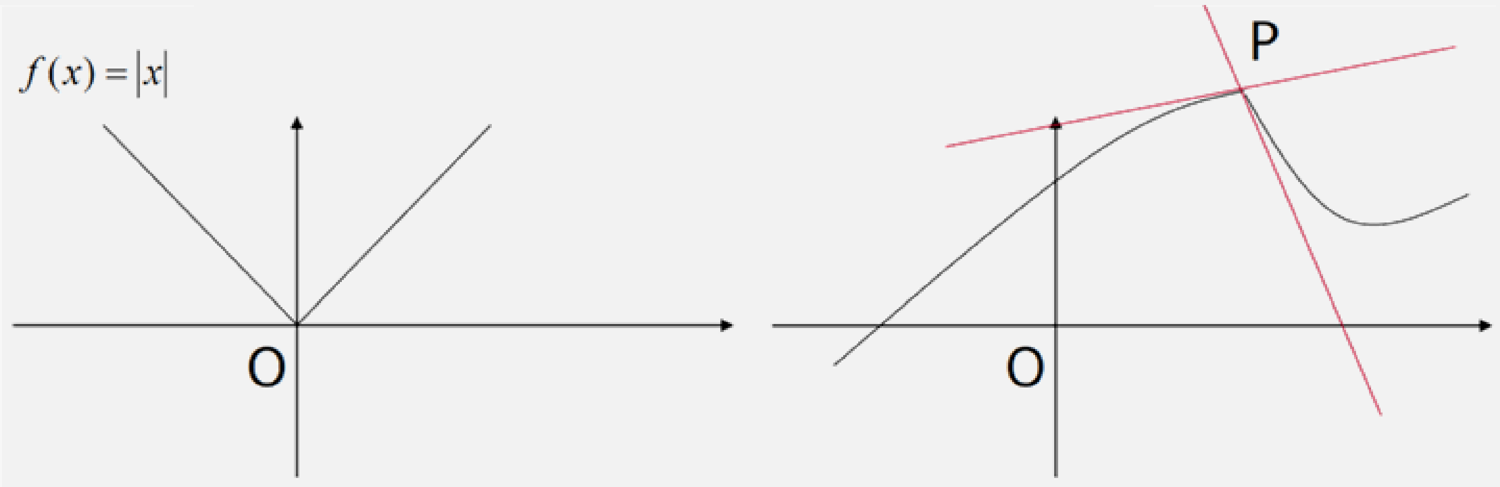
The function $f(x) = |x|$ is not differentiable in the point $x_0 = 0$ (even if it is continuous in that point) and the point $x_0$ is called **corner point**

# Definition: corner point

Let $f(x)$ be a function defined on an interval $[a, b]$, and let $x_0$ be an interior point of $[a, b]$.

If $f$ admits at $x_0$ a finite right derivative and a finite left derivative, but these derivatives are different, then $f$ is not differentiable at $x_0$, and $x_0$ is called a corner point.

From the graphical point of view, at a corner point $x_0$ the graph admits two tangent lines (from the right and from the left) that are not parallel to the $y$-axis.



## Definition: corner point

In general, any function that contains an absolute value in its analytic expression is not differentiable at the points $x$ where the argument of the absolute value vanishes.
Such points are corner points

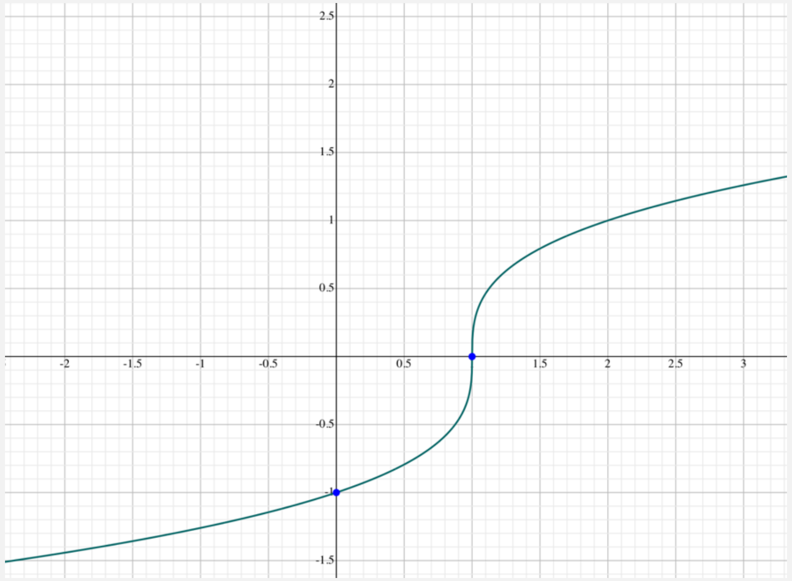
# Function not differentiable at a point: inflection with vertical tangent

$$f(x) = \sqrt[3]{x-1}$$

This function is defined and continuous on all of $\mathbb{R}$; in particular, it is defined and continuous at $x_0 = 1$.

Let us verify differentiability at $x_0 = 1$.

Compute the difference quotient in the case $x_0 = 1$:




$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\sqrt[3]{1+h-1} - \sqrt[3]{1-1}}{h} = \frac{\sqrt[3]{h}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0} h^{\left(\frac{1}{3}-1\right)} = \lim_{h\to 0} h^{-\frac{2}{3}} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty$$

Hence:

$$\lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = +\infty \qquad \lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = +\infty$$

# Function not differentiable at a point: inflection with vertical tangent

The function $f(x) = \sqrt[3]{x-1}$ admits in the point $x_0 = 1$ a right and a left derivative that are equal but not finite:

$$f'_{+}(x_0) = +\infty \text{ and } f'_{-}(x_0) = +\infty$$

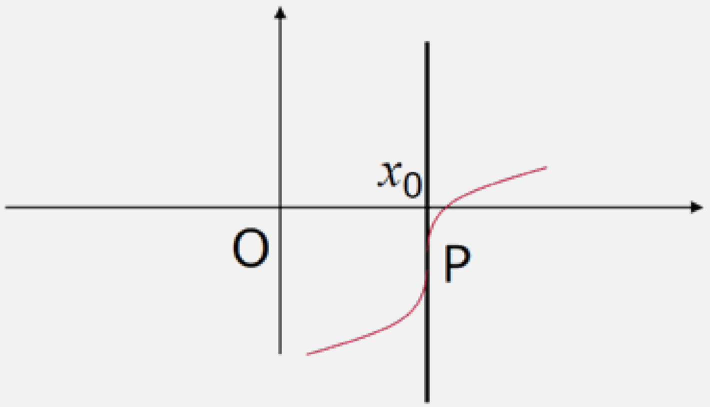
⇓

The function $f(x) = \sqrt[3]{x-1}$ is not differentiable in the point $x_0 = 1$ (even if it is continuous there), and the point $x_0$ is called **inflection with a vertical tangent**

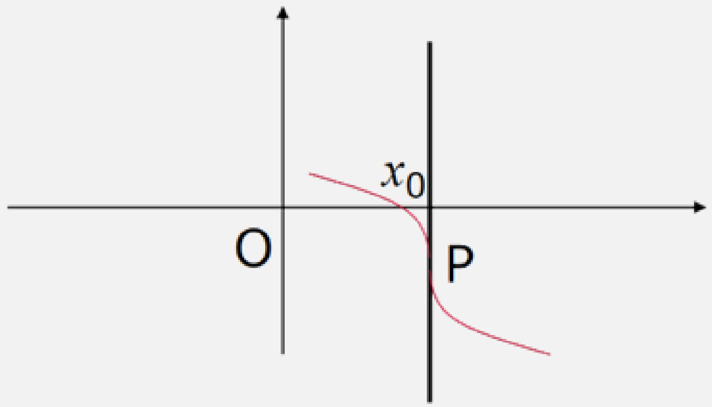
# Definition: inflection with vertical tangent

Let $f(x)$ be a function defined on an interval $[a, b]$ , and let $x_0$ be an interior point.
If $f$ admits at $x_0$ equal right and left limits of the difference quotient, but both are infinite (i.e., both $+\infty$ or both $-\infty$), then $f$ is not differentiable at $x_0$ , and $x_0$ is called an **inflection point with vertical tangent**.

Graphically, at such a point (inflection point with a vertical tangent) the graph admits a tangent line parallel to the $y$-axis.




$$\lim_{h\to 0^{\pm}} \frac{f(x_0+h)-f(x_0)}{h} = +\infty$$




$$\lim_{h\to 0^{\pm}} \frac{f(x_0+h)-f(x_0)}{h} = -\infty$$

# Function not differentiable at a point: <u>cusp</u>

$$f(x) = \sqrt[3]{|x|}$$

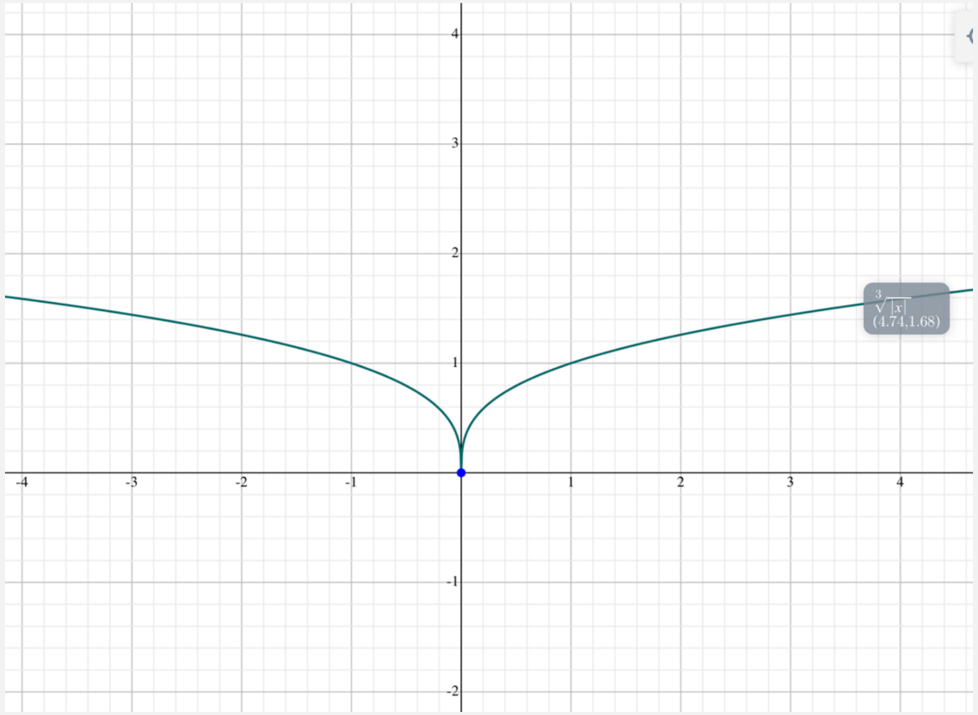
This function is defined and continuous on all of $\mathbb{R}$; in particular, it is defined and continuous at $x_0 = 0$.

Let us verify if the function is also differentiable at $x_0 = 0$.

Compute the difference quotient (with $x_0 = 0$):

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{\sqrt[3]{|h|} - 0}{h} = \frac{\sqrt[3]{|h|}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt[3]{|h|}}{h} = \begin{cases} \lim_{h\to 0^+} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0^+} \frac{1}{h^{\frac{2}{3}}} = \lim_{h\to 0^+} \frac{1}{\sqrt[3]{x^2}} = +\infty \\ \lim_{h\to 0^-} \frac{\sqrt[3]{-h}}{h} = \lim_{h\to 0^-} -\frac{1}{h^{\frac{2}{3}}} = \lim_{h\to 0^-} -\frac{1}{\sqrt[3]{x^2}} = -\infty \end{cases}$$



# Function not differentiable at a point: cusp

Hence:

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = +\infty \qquad \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = -\infty$$

Thus the right and left limits are different and not finite.

$f'_+(x_0) = +\infty$ and $f'_-(x_0) = -\infty$

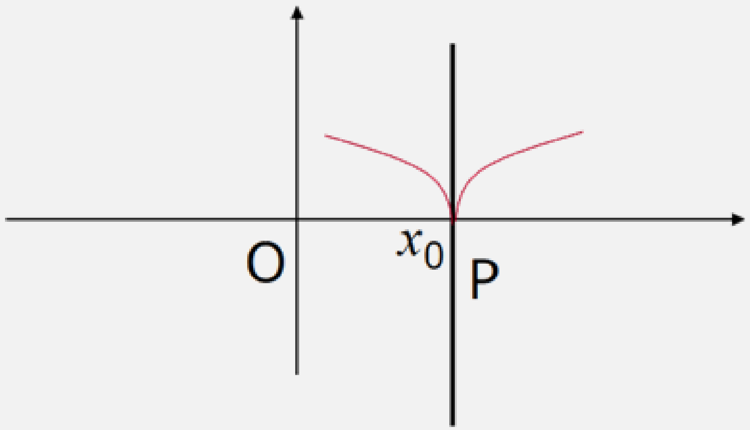
⇓

The function $f(x) = \sqrt[3]{|x|}$ is not differentiable in the point $x_0 = 0$ (even if it is continuous there), and the point $x_0$ is called **cusp**

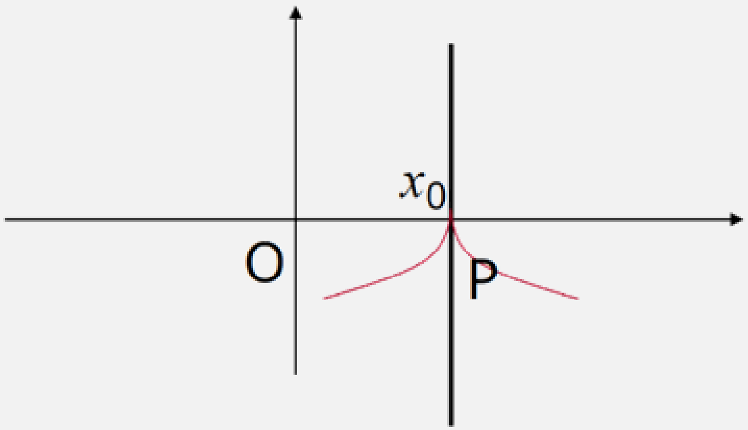
## Definition: cusp

Let $f(x)$ be defined on $[a, b]$ and let $x_0$ be an interior point.
If $f$ admits at $x_0$ right and left limits of the difference quotient that are different and infinite (one equal to $+\infty$ and the other equal to $-\infty$, or vice versa), then $f$ is not differentiable at $x_0$, and $x_0$ is called a **cusp**.

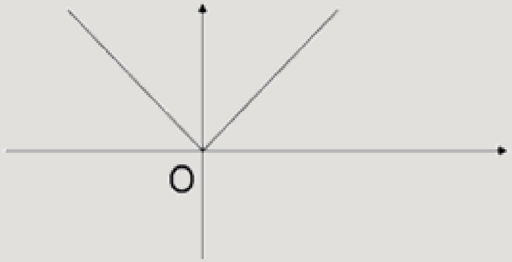
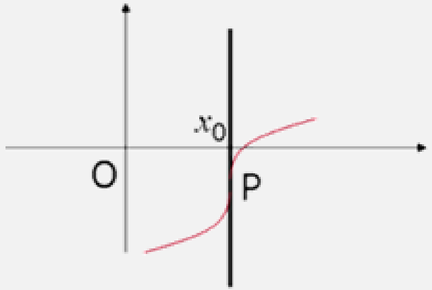
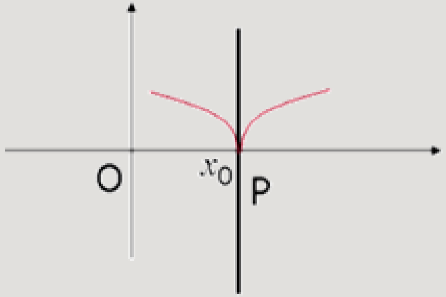
Graphically, at a cusp the graph admits a tangent line parallel to the $y$-axis.




$$\lim_{x \to x_0^{\pm}} \frac{f(x) - f(x_0)}{x - x_0} = \pm\infty$$




$$\lim_{x \to x_0^{\pm}} \frac{f(x) - f(x_0)}{x - x_0} = \mp\infty$$

| Type of point | Definition | Graph |
|---|---|---|
| **Corner point** | If $f$ admits in $x_0$ right and left derivative, both finite but distinct |   |
| **Inflection with a vertical tangent** | If $f$ admits in $x_0$ right and left limits of the difference quotient that are equal but infinite (that is both equal to $+\infty$ or $-\infty$) |   |
| **Cusp** | If $f$ admits in $x_0$ right and left limits of the difference quotient that are different and infinite (that is one equal to $+\infty$ and one to $-\infty$ or vice versa) |   |

- **CHARACTERIZATION OF MONOTONE FUNCTIONS**

# Monotone Functions: Increasing

Let

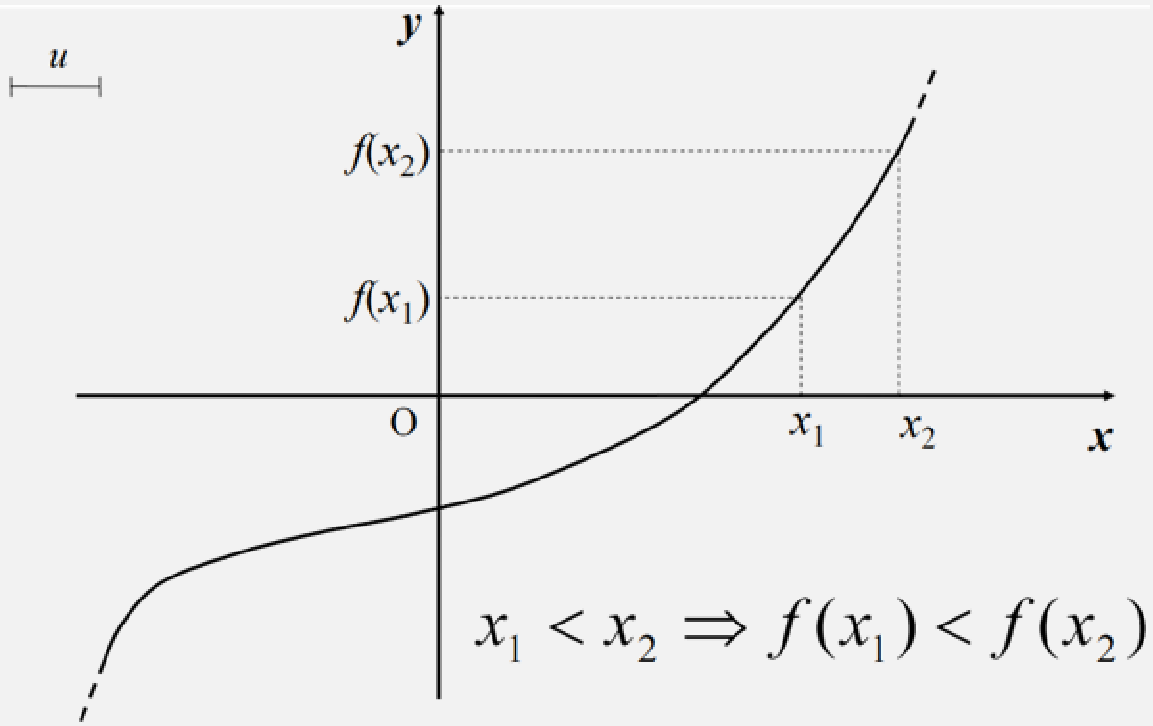
$f : A \to B,$ with $A, B \subseteq \mathbb{R},$ $A, B \neq \emptyset$

$f$ is **strictly increasing** in $A$ if:

$$\forall x_1, x_2 \in A : x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

$f$ is **increasing** in $A$ if:

$$\forall x_1, x_2 \in A : x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$



# Monotone Functions: Decreasing

Let

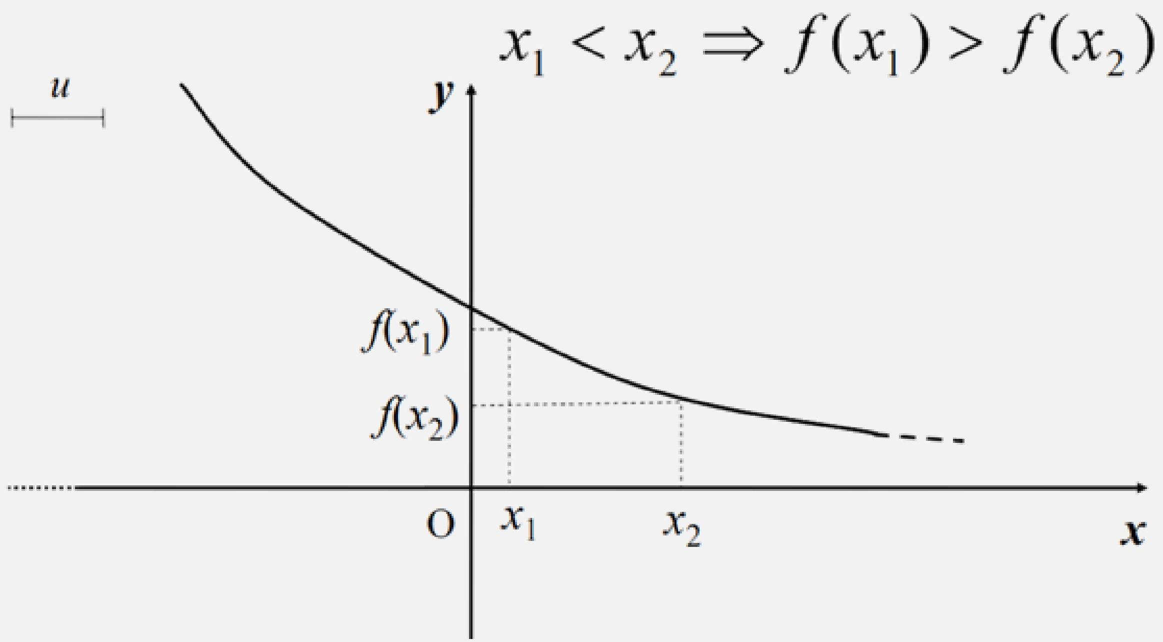
$f : A \to B,$ with $A, B \subseteq \mathbb{R},$ $A, B \neq \emptyset$

$f$ is **strictly decreasing** in $A$ if:

$$\forall x_1, x_2 \in A : x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

$f$ is **decreasing** in $A$ if:

$$\forall x_1, x_2 \in A : x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$



# Definition (increasing/decreasing)

We now examine how the sign of the first derivative characterizes monotonicity.

Let $f : [a, b] \to \mathbb{R}$ be differentiable at interior points, and let $x_0, x_1 \in (a, b)$.

We already know:

- $f$ increasing if : $x_1 > x_0 \Rightarrow f(x_1) \geq f(x_0)$

  $$x_0 + h > x_0 \Rightarrow f(x_0 + h) \geq f(x_0)$$

- $f$ decreasing if : $x_1 > x_0 \Rightarrow f(x_1) \leq f(x_0)$

  $$x_0 + h > x_0 \Rightarrow f(x_0 + h) \leq f(x_0)$$

Hence:

- $f$ increasing $\Leftrightarrow \frac{f(x_0+h)-f(x_0)}{x_0+h-x_0} \geq 0$

- $f$ decreasing $\Leftrightarrow \frac{f(x_0+h)-f(x_0)}{x_0+h-x_0} \leq 0$

⇓ *Passing to the limit as $x \to x_0$*

- $f$ increasing $\Leftrightarrow f'(x_0) \geq 0, \forall x_0 \in (a, b)$

- $f$ decreasing $\Leftrightarrow f'(x_0) \leq 0, \forall x_0 \in (a, b)$

Hence:

## Monotonicity criterion

Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on $(a, b)$.
Then:

- $f$ increasing on $[a, b] \Leftrightarrow f'(x) \geq 0, \forall x \in (a, b)$

- $f$ decreasing on $[a, b] \Leftrightarrow f'(x) \leq 0, \forall x \in (a, b)$

From the monotonicity criterion, it is possible to study relative and absolute maxima and minima of a function.

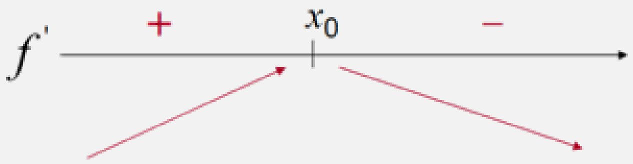
Let $f : [a, b] \to \mathbb{R}$ be defined on $[a, b]$ and differentiable on $(a, b)$.

To find relative and absolute extrema (if they exist), one may proceed as follows:

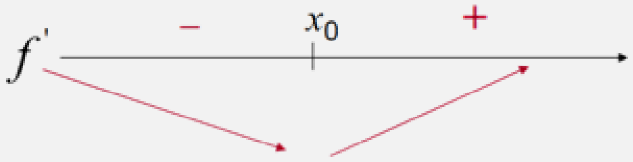
1. Compute $f(a)$ and $f(b)$

2. Determine the derivative $f'(x)$ and solve $f'(x) = 0$. The solutions are stationary points; among them may lie local extrema within $(a, b)$.
3. If:
   a. $f'(x) = 0$ has no solutions, then $f(a)$ and $f(b)$ (if distinct) are absolute extrema.
   *b.* $f'(x) = 0$ has solutions and, for instance, $x = x_0$ is stationary, then to decide whether $x_0$ is a local extremum one studies the sign of $f'(x)$ in a neighborhood of $x_0$ and applies the monotonicity criterion.

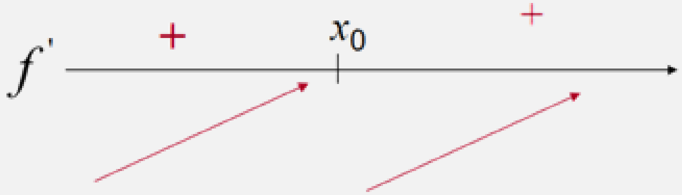
More precisely, if $f'(x_0) = 0$ , one can verify that:



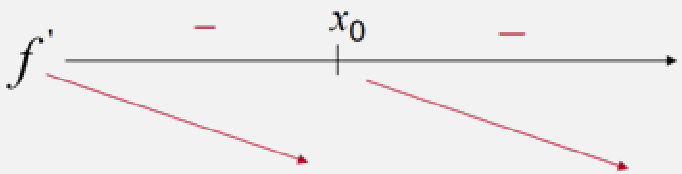

$x_0$ point of local maximum




$x_0$ point of local minimum




$x_0$ point of inflection




$x_0$ point of inflection

4. Having found possible local extrema, compute $f$ at those points and compare with $f(a)$ and $f(b)$.

As a consequence of the monotonicity criterion:

## Characterization of constant functions on an interval

Let $f$ be a function differentiable on $[a, b]$ and suppose that $f'(x) = 0, \forall x \in (a, b)$

$$\Updownarrow$$

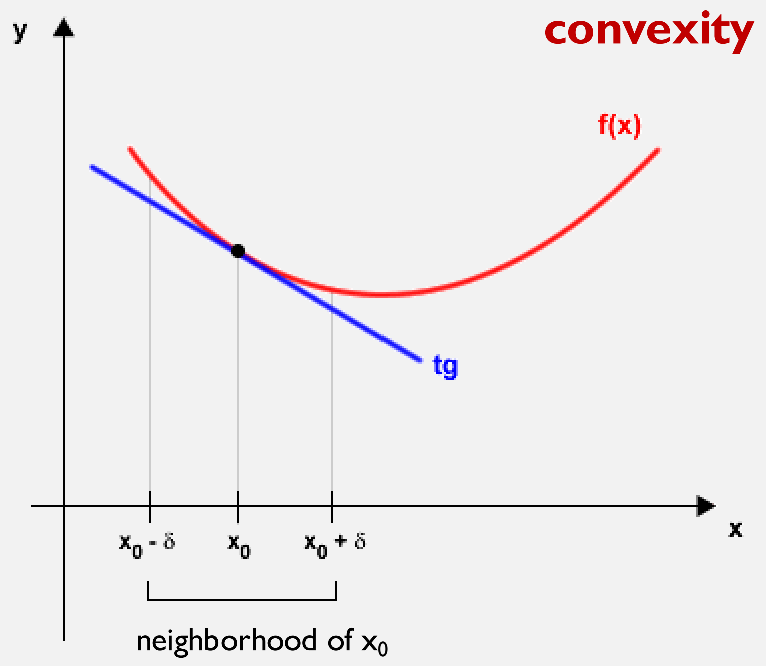
then, $f$ is constant on $[a, b]$

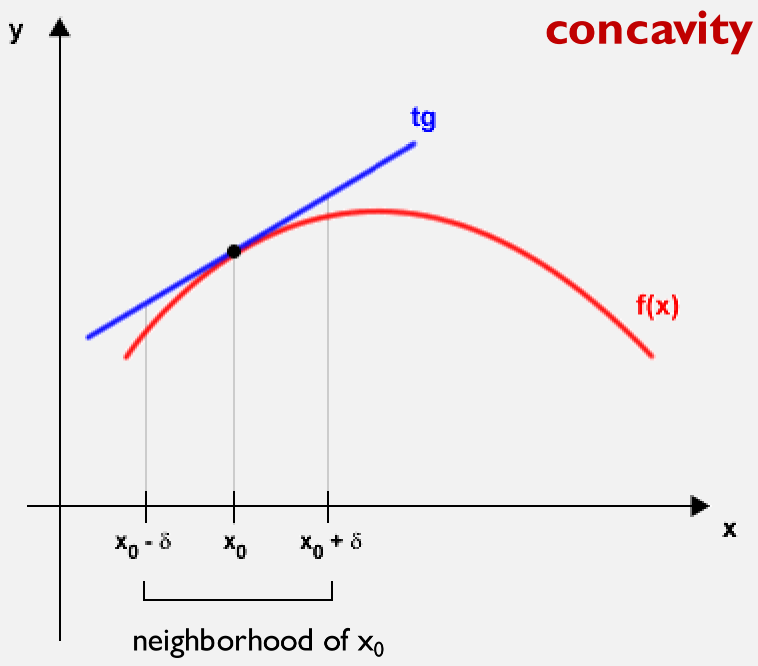
- **CONCAVITY AND CONVEXITY**

# Convex and concave functions

A function $f(x)$ is said to be **convex** on $[a, b]$ if $\forall x_0 \in [a, b]$ , the graph of $f(x)$ lies **above** the tangent line to the graph at the point $(x_0, f(x_0))$.

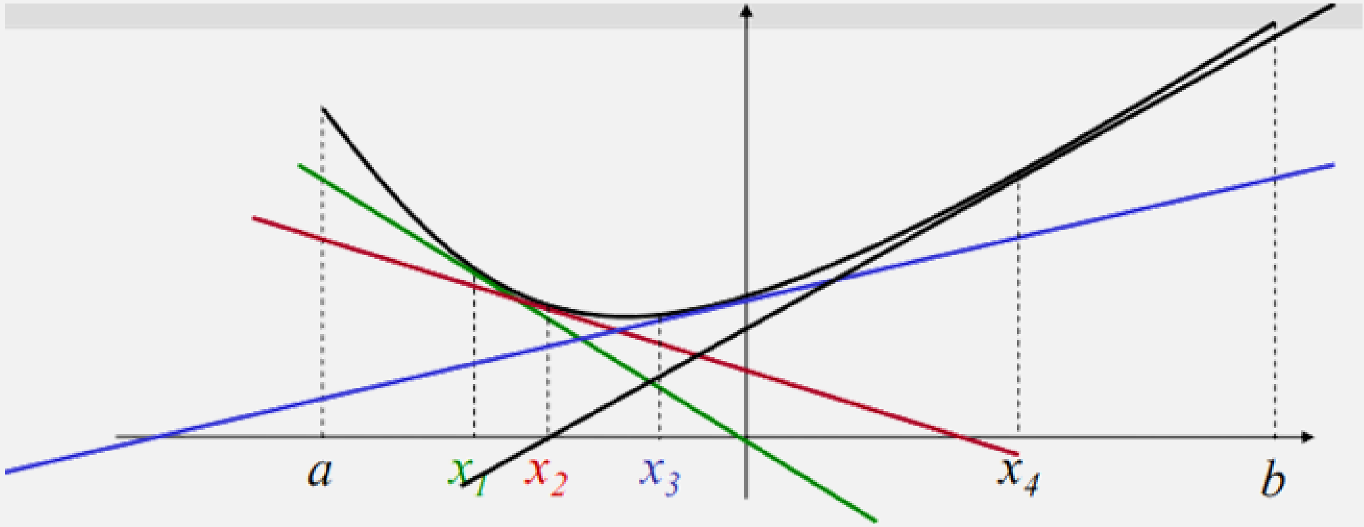
A function $f(x)$ is said to be **concave** on $[a, b]$ if $\forall x_0 \in [a, b]$ , the graph of $f(x)$ lies **below** the tangent line to the graph at the point $(x_0, f(x_0))$.



# Geometric meaning of convexity

As $x_0$ increases in $[a, b]$ from smaller to larger values, the slope of the tangent line to the graph of $f$ at $(x_0, f(x_0))$ changes; in particular, the **slope increases**, passing from smaller values (possibly negative) to larger values.

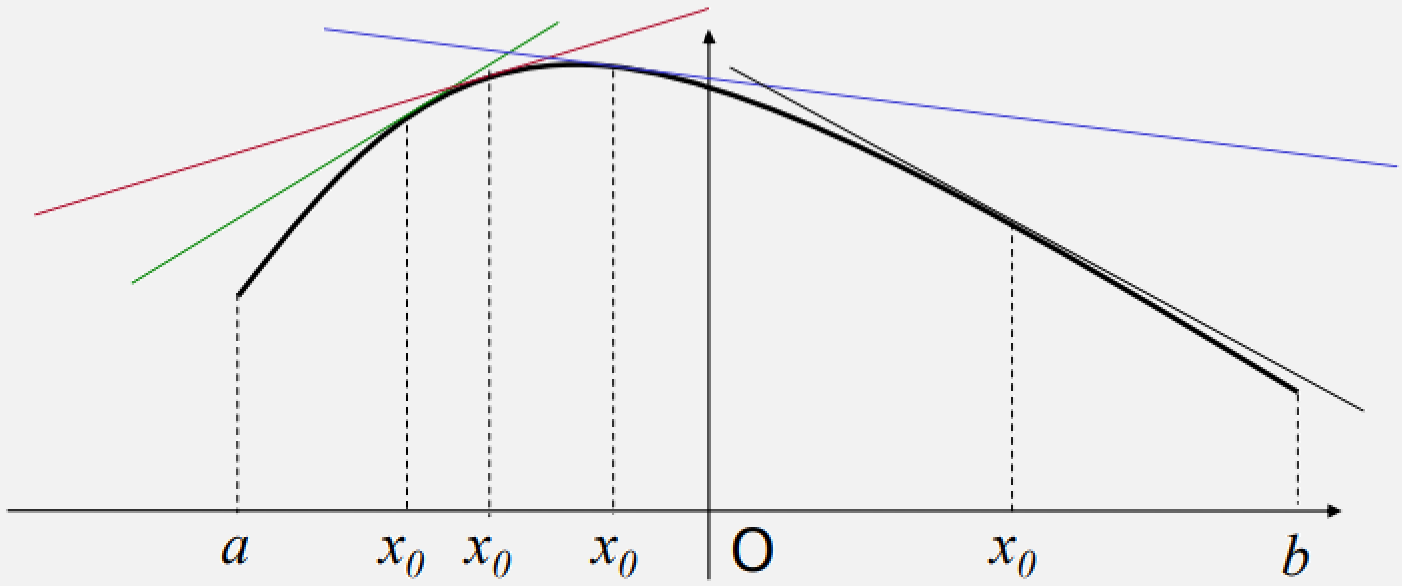



$x_1 < x_2 \Rightarrow f'(x_1) < f'(x_2) \Rightarrow f'(x)$ increasing

$f$ convex in [a,b] $\Leftrightarrow f'$ increasing in $[a,b] \Leftrightarrow (f')' \geq 0$ in $[a,b] \Leftrightarrow f'' \geq 0$ in $[a,b]$

# Geometric meaning of concavity

As $x_0$ increases in $[a, b]$ from smaller to larger values, the slope of the tangent line to the graph of $f$ at $(x_0, f(x_0))$ changes; in particular, the **slope decreases**, passing from larger values to smaller ones (possibly negative) .



# Convexity criterion

Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on $(a,b)$.

Then:

- $f$ convex on $[a,b] \Leftrightarrow f'$ increasing on $[a,b] \Leftrightarrow f'' \geq 0$ on $[a,b]$

- $f$ concave on $[a,b] \Leftrightarrow f'$ decreasing on $[a,b] \Leftrightarrow f'' \leq 0$ on $[a,b]$

If the first derivative of a function is increasing on $[a,b]$, the function $f(x)$ is convex
If the first derivative of a function is decreasing on $[a,b]$, the function $f(x)$ is concave

*If the first derivative represents the slope of the graph of the function, the second derivative represents the variation of that slope.*