- **THE STUDY OF A FUNCTION**

# Outline

$$f(x) = \frac{x^3}{x^2 - 4}$$

## 1. DOMAIN OF $f$

The first important piece of information is to determine for which real values the function is defined.
In this example, the domain is given by $D_f = \mathbb{R} - \{\pm 2\}$, that is, it consists of all real numbers except those that make the denominator equal to zero ($\pm 2$).

Recall:

- For rational functions, the domain is given by those points for which the denominator is nonzero;
- For algebraic functions involving even roots, the domain consists of those values for which the radicand is non-negative;
- For functions involving a logarithm, the argument must be positive;
- For exponential, trigonometric, and polynomial functions, the domain coincides with the whole real line.

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## 2. INTERSECTIONS WITH THE AXES

Determining the points of intersection between the curve and the coordinate axes means finding:

- the coordinates of the points where the function intersects the $x$-axis, and
- the coordinates of the points where the function intersects the $y$-axis.

- To find the points of intersection with the $y$-axis, we evaluate the function at $x = 0$.
- To find the points of intersection with the $x$-axis, we determine the values of $x$ for which $y = 0$.

In this example, the function intersects both the $x$-axis and the $y$-axis only at the origin.

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## 3. SIGN OF $f$

We want to determine the values of $x$ for which $f(x) \geq 0$. In the example, we need to study separately the sign of the numerator and of the denominator, and then combine the obtained information. The numerator is positive or null for $x \geq 0$, while the denominator is positive for $x < -2$ and $x > 2$. Hence, we obtain that $f(x) \geq 0$ for $-2 < x \leq 0, x > 2$.

Recall:

- Studying the sign of a rational function amounts to solving two inequalities, one for the numerator and one for the denominator, and then combining the results;
- An exponential function is always strictly positive;
- To study the sign of a logarithmic function, it is necessary to determine for which values of $x$ the argument of the logarithm is greater than or equal to $1$;
- For odd roots, the sign is determined by the sign of the radicand;
- Even roots are always non-negative;
- For trigonometric functions, it is useful to refer to the unit circle to determine their sign.

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## 4. ASYMPTOTES OF $f$

Recall:

- if $\lim_{x \to x_0^+} f(x) = \pm\infty$, then, the line $x = x_0$ is called vertical asymptote for $f(x)$
- if $\lim_{x \to +\infty} f(x) = l_1$, $\lim_{x \to -\infty} f(x) = l_2$, then $f(x)$ admits two horizontal asymptotes: $y = l_1$ to $+\infty$ and $y = l_2$ to $-\infty$

- If the following conditions hold, then the graph of the function $f(x)$ admits oblique asymptote of equation $y = mx + q$: $\lim_{x \to \pm\infty} f(x) = \pm\infty$ , $\lim_{x \to \pm\infty} \frac{f(x)}{x} = m \neq 0\ finite$ , $\lim_{x \to \pm\infty} [f(x) - mx] = q\ finite$

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We compute the limit of $f$ as $x \to \pm\infty$ in the case where the domain contains arbitrarily large positive and negative values, and then we study the one-sided limits at the boundary points of the domain (points that do not belong to the domain, but such that every open interval around them intersects the domain of $f$).

In the example, the critical points are $x = \pm 2$, and it is necessary to compute separately the left-hand and right-hand limits, since the function changes sign at these points:

$$\lim_{x \to -2^-} \frac{x^3}{x^2 - 4} = -\infty, \ \lim_{x \to -2^+} \frac{x^3}{x^2 - 4} = +\infty$$

$$\lim_{x \to 2^-} \frac{x^3}{x^2 - 4} = -\infty, \ \lim_{x \to 2^+} \frac{x^3}{x^2 - 4} = +\infty$$

When, as in this case, at least one of the one-sided limits as $\mathrm{x} \to a \in \mathbb{R}$ is equal to $\pm\infty$, we say that the function has a vertical asymptote $x = a$. Therefore, the function in this example has vertical asymptotes at $x = -2$ and $x = 2$.

Oblique asymptotes:

In this example, $\lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{x^3}{(x^2-4)x} = 1 = m$

Moreover, $\lim_{x\to\pm\infty}[f(x) - mx] = \lim_{x\to\pm\infty}[\frac{x^3}{(x^2-4)} - x] = \lim_{x\to\pm\infty}\frac{x^3-x^3+4x}{(x^2-4)} = 0$

Therefore, $y = x$ is the oblique asymptote.

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## 5. MAXIMA AND MINIMA

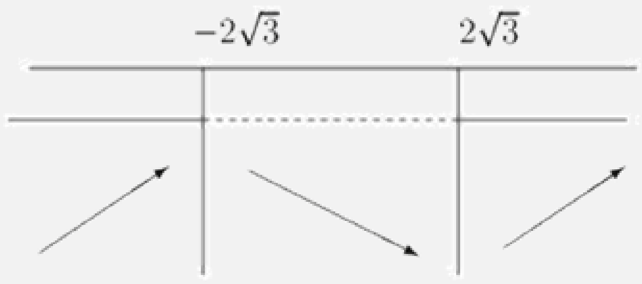
To compute maxima and minima of a function, it is necessary to compute the first derivative and study its sign.

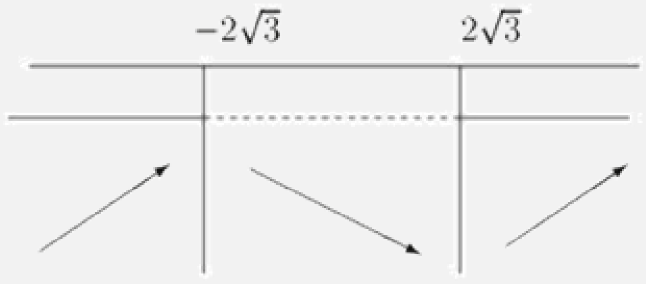
$$f'(x) = \frac{x^2(x^2 - 12)}{(x^2 - 4)^2} \geq 0 \Rightarrow x \leq -2\sqrt{3}, x \geq 2\sqrt{3}$$

Hence, in intervals where $f'(x) > 0$ the function is increasing, while in intervals where $f'(x) < 0$ the function is decreasing.

The derivative vanishes at $x = \pm 2\sqrt{3}$. To determine whether these points are maxima, minima, or points of inflection, we study the sign of the derivative. Thus:

- the function is increasing on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$
- the function is decreasing on $(-2\sqrt{3}, 2\sqrt{3})$






The points at which the derivative vanishes $x = \pm 2\sqrt{3}$. The point $x = -2\sqrt{3}$ is a local minimum, while $x = 2\sqrt{3}$ is a local maximum.

We also observe the presence of a point of inflection at $x = 0$, that is, a point at which the first derivative vanishes but which is neither a maximum nor a minimum.
Therefore, at $x = 0$, the graph of $f$ has a horizontal tangent line.

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## 6. CONCAVITY

Studying the sign of the second derivative $f''(x)$, that is, the derivative of the first derivative, allows us to analyze the concavity of the function $f$. in fact, if $f''(x) > 0$, the function is convex; if $f''(x) < 0$, the function is concave.

In this example: $f''(x) = \frac{8x(x^2+12)}{(x^2-4)^3} \geq 0 \rightarrow -2 < x \leq 0, x > 2$

If $f''(x) > 0$, the function $f'(x)$ is increasing, hence the tangent line to the graph has an increasing slope; if $f''(x) < 0$, the function $f'(x)$ is decreasing, hence the tangent line to the graph has a decreasing slope;

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## 7. **GRAPH**

At this point, it is necessary to combine all the information obtained in the previous steps and draw the graph of the function $f$.