(1) FINDING THE DOMAIN

$$f(x) = \frac{3x-2x^3}{x^4-1} \longrightarrow$$ FIND THE POINTS WHERE THE DENOMINATOR VANISHES

$$\Downarrow$$

$$x^4-1 \neq 0 \longrightarrow (x^2+1)(x^2-1) \neq 0$$

$$x^2+1 \neq 0 \qquad\qquad x^2-1 \neq 0$$

$$\Downarrow \qquad\qquad \searrow$$

$$x^2 \neq -1 \quad \forall x \in \mathbb{R} \qquad\qquad x \neq +1, \; x \neq -1$$

A

B

$$D = \{x \in \mathbb{R} : x \neq \pm 1\} = \mathbb{R} - \{-1\} - \{+1\}$$

C

$$= (-\infty, -1) \cup (-1, +1) \cup (+1, +\infty)$$

(2) $$f(x) = \frac{1}{x-2} \longrightarrow$$ DENOMINATOR $\neq 0$

$$x-2 \neq 0 \longrightarrow x \neq +2$$

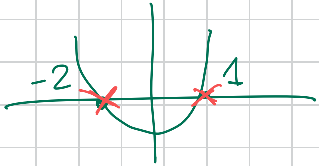
$$D = \{x \in \mathbb{R} : x \neq +2\} = (-\infty, +2) \cup (+2, +\infty) = \mathbb{R} - \{+2\}$$

③ FINDING THE DOMAIN

$$f(x) = \sqrt{x^2+3} \longrightarrow \text{radicand of square root} \geq 0$$

$$x^2+3 \geq 0 \rightarrow x^2 \geq -3 \rightarrow \text{always true!} \quad \forall x \in \mathbb{R}$$

$$\boxed{D = \mathbb{R}}$$



④ FINDING THE DOMAIN

$$f(x) = \frac{4-5x^2}{x^2+x-2} \longrightarrow x^2+x-2 \neq 0 \rightarrow \begin{array}{l} x \neq -2 \\ x \neq +1 \end{array}$$

Ⓐ Ⓑ

$$D = \{x \in \mathbb{R} : x \neq -2,\ x \neq +1\} = \mathbb{R} - \{-2\} - \{+1\} =$$

Ⓒ $(-\infty, -2) \cup (-2, +1) \cup (+1, +\infty)$

(5) FINDING THE DOMAIN

$f(x) = \sqrt{x-2} \longrightarrow$ radicand of square root ≥ 0

$x-2 \geq 0 \rightarrow x \geq +2$

$D = \{x \in \mathbb{R} : x \geq +2\}$

(6) FINDING THE DOMAIN

$f(x) = \sqrt{|x|-2} \longrightarrow$ radicand of square root ≥ 0

$$|x|-2 \geq 0 \longrightarrow |x| \geq +2 \longrightarrow \begin{cases} \text{if } x \geq 0 \rightarrow x \geq 2 \\ \text{if } x < 0 \rightarrow -x \geq 2 \end{cases}$$

$-x \geq 2 \downarrow x \leq -2$

⑦ FINDING THE DOMAIN

$f(x) = \sqrt{\log_e(x) + 1}$ $\longrightarrow$ 1) radicand of square root ≥ 0

2) argument of log > 0

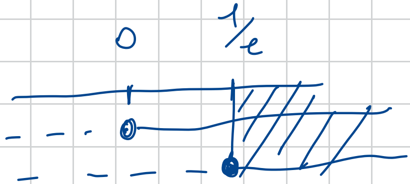
D?

$\downarrow$

$$\begin{cases} \log_e x + 1 \geq 0 \rightarrow \log_e x \geq -1 \rightarrow \cancel{e}^{\cancel{\log_e} x} \geq e^{-1} \\ x > 0 \end{cases}$$

$\downarrow$

$x \geq e^{-1} \rightarrow x \geq \frac{1}{e}$



$D = \left\{ x \in \mathbb{R} : x \geq \frac{1}{e} \right\}$

$\sqrt{\log \frac{1}{x^2}}$ $\longrightarrow$ 1) radicand of sq. root ≥ 0

2) argum. of log > 0

3) denom. $\neq 0$

$$\begin{cases} \log \frac{1}{x^2} \geq 0 & \text{(radicand)} \\ 1/x^2 > 0 & \text{(argum.)} \\ x^2 \neq 0 & \text{(denom.)} \end{cases}$$

8 FIND THE DOMAIN

1) denominator $\neq 0$

2) square root $\to$ radic ≥ 0

$$f(x) = \frac{x+1}{\sqrt{x+2} - \sqrt[3]{4x+5}}$$

$$\sqrt{x+2} - \sqrt[3]{4x+5} \neq 0 \longrightarrow \sqrt{x+2} \neq \sqrt[3]{4x+5}$$ 1

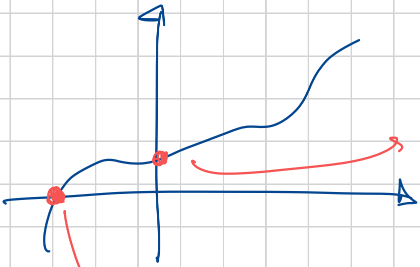
condition of existence

2

$$x+2 \geq 0 \to x \geq -2$$

$$D = \left\{ x \in \mathbb{R} : x \geq -2 , \sqrt{x+2} \neq \sqrt[3]{4x+5} \right\}$$

(1) FINDING THE INTERSECTIONS WITH AXES

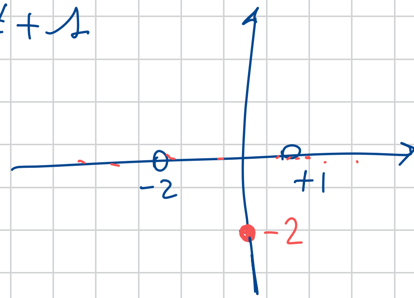


inters. with y-axis $(x=0)$

inters. with x-axis $(y=0)$

$$f(x) = \frac{1-5x^2}{x^2+x-2}$$

D? $x \neq -2$, $x \neq +1$



inters. with y-axis $(x=0)$:

$$\begin{cases} x=0 \\ y = \dfrac{4-5x^2}{x^2+x-2} \end{cases} \qquad \begin{cases} x=0 \\ y = \dfrac{4-0}{0+0-2} = -2 \end{cases} \longrightarrow A(0,-2)$$

inters. with x-axis $(y=0)$:

$$\begin{cases} y=0 \\ y = \dfrac{4-5x^2}{x^2+x-2} \end{cases} \longrightarrow \frac{4-5x^2}{x^2+x-2} = 0 \longrightarrow 4-5x^2=0 \longrightarrow x = \pm\frac{2}{\sqrt{5}}$$

$$\longrightarrow x^2+x-2=0 \text{ NEVER! (D)}$$

$$B\left(-\frac{2}{\sqrt{5}}, 0\right) \qquad C\left(+\frac{2}{\sqrt{5}}, 0\right)$$

① LIMITS

$$\lim_{x\to+\infty}\left(x^3+x^2+2\right) = +\infty+\infty+2 = +\infty$$

↳ continuous function → $f(+\infty)$

②
$$\lim_{x\to-1}\frac{x^2-1}{x+1} = \frac{+1-1}{-1+1} = \frac{0}{0} \rightarrow \text{INDET. FORM!}$$

factorize the limit

$$\lim_{x\to-1}\frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}} = \lim_{x\to-1}(x-1) = -1-1 = -2 \checkmark$$

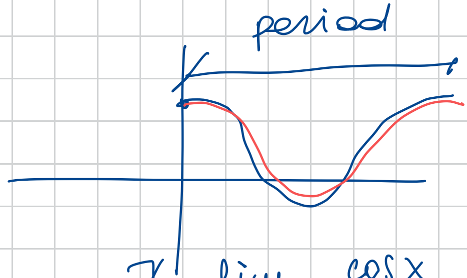
③
$$\lim_{x\to+\infty}\frac{x^3+x^2-1}{2x^2+2} = \frac{+\infty+\infty-1}{+\infty+2} = \frac{\infty}{\infty} \rightarrow \text{INDET. FORM!}$$

factorize the limit!

$$\lim_{x\to+\infty}\frac{\cancel{x^3}\left(1+\frac{1}{x}-\frac{1}{x^3}\right)}{\cancel{x^3}\left(\frac{2}{x}+\frac{2}{x^3}\right)} = \frac{1+\cancel{\frac{1}{\infty}}^{\,0}-\cancel{\frac{1}{\infty}}^{\,0}}{{}_{0}\,\cancel{\frac{2}{\infty}}+\cancel{\frac{2}{\infty}}^{\,0}} = \frac{+1}{0} = +\infty \checkmark$$

④ $\displaystyle\lim_{x\to+\infty} \frac{x+2+\cos x}{3-x}$

↳ $\displaystyle\frac{+\infty+2+\cos(+\infty)}{3-\infty}$



$\displaystyle\nexists \lim_{x\to\pm\infty} \cos x$

let's avoid this non-sense limit

$$\lim_{x\to+\infty}\frac{\cancel{x}\left(1+\frac{2}{x}+\frac{\cos x}{x}\right)}{\cancel{x}\left(\frac{3}{x}-1\right)} = \frac{1+\cancelto{0}{\frac{2}{+\infty}}+\cancelto{0}{\frac{\cos(+\infty)}{+\infty}}}{\cancelto{0}{\frac{3}{+\infty}}-1} = \frac{1}{-1} = -1 \checkmark$$

⑤ $$\lim_{x\to+\infty}\frac{x^2+\sin e^x}{2x} \to \lim_{x\to+\infty}\frac{\cancel{x^2}\left(1+\cancelto{0}{\frac{\sin e^x}{x^2}}\right)}{2\cancel{x}}$$

$$= \frac{+\infty(1+0)}{2} = +\infty \checkmark$$

FINDING ASYMPTOTES

IF $\lim_{x \to \pm\infty} f(x) = K \longrightarrow$ HORIZ. AS.

endpoints of my DOMAIN

IF $\lim_{x \to c} f(x) = \pm\infty \longrightarrow$ VERT. AS.

IF $\lim_{x \to \pm\infty} f(x) = \pm\infty \longrightarrow$ NO HORIZ. AS.,

maybe OBLIQUE AS.

$$y = mx + q$$

other conditions:

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x} \longrightarrow \text{FINITE AND } \neq 0$$

$$q = \lim_{x \to \pm\infty} [f(x) - mx]$$

$$f(x)=\sqrt{x^2+1}$$

$$\lim_{x\to\pm\infty}\sqrt{x^2+1}=\sqrt{+\infty+1}=+\infty \longrightarrow$$ NO HORIZ. AS.

OBLIQUE?

$$m=\lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{x}=\lim_{x\to+\infty}\sqrt{\frac{x^2+1}{x^2}}$$

$$=\lim_{x\to+\infty}\sqrt{1+\frac{1}{x^2}}=\sqrt{1+\frac{1}{\infty}}\,0=+1$$

Finite and $\neq 0$ ✓

$\lim[f(x)-mx]$

$$q=\lim_{x\to+\infty}\sqrt{x^2+1}-1\cdot x=+\infty-\infty$$ INDET. FORM!

$$\lim_{x\to+\infty}\sqrt{x^2+1}-x\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\lim_{x\to+\infty}\frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}\quad+x}$$

$$=\frac{+1}{+\infty+\infty}=\frac{1}{\infty}=0^+$$

$x\to-\infty \longrightarrow m=-1$

$q=0$

NO HORIZ.

2 OBL. $\to y=x,\ y=-x$

STUDY OF A FUNCTION

$$f(x) = \frac{x^2-1}{x-2}$$

① DOMAIN $\longrightarrow$ $x-2 \neq 0$ $\quad x \neq +2$ $\quad D = \{x \in \mathbb{R} : x \neq 2\}$

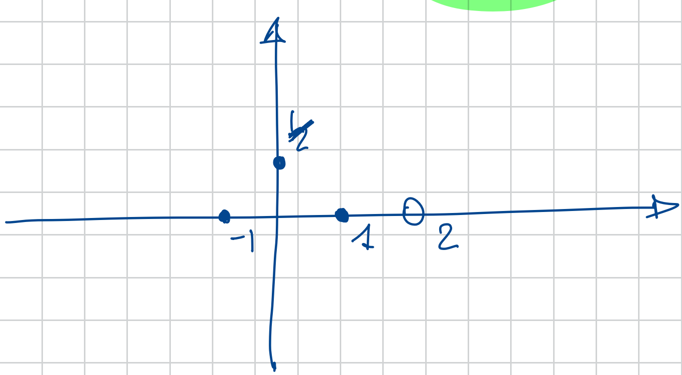
② INTERSECTIONS

Y AXIS $\longrightarrow$ $\begin{cases} x = 0 \\ y = \dfrac{x^2-1}{x-2} \end{cases}$ $\quad \begin{cases} x = 0 \\ y = \dfrac{0-1}{0-2} = \dfrac{1}{2} \end{cases}$ $\rightarrow A\left(0, \frac{1}{2}\right)$

X AXIS $\longrightarrow$ $\begin{cases} y = 0 \\ y = \dfrac{x^2-1}{x-2} \end{cases}$ $\longrightarrow \dfrac{x^2-1}{x-2} = 0 \longrightarrow x^2-1=0$

$x = \pm 1$

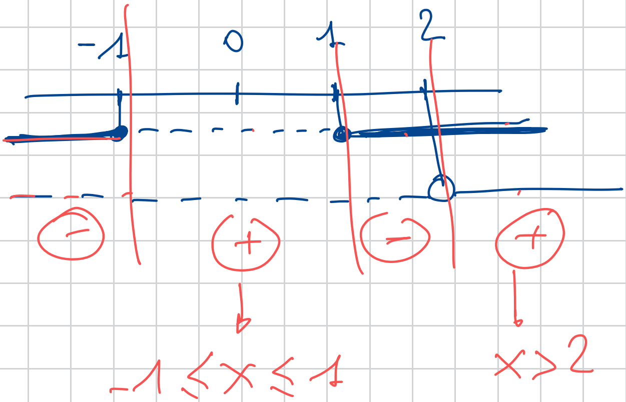
$B(-1, 0)$ $\quad C(+1, 0)$

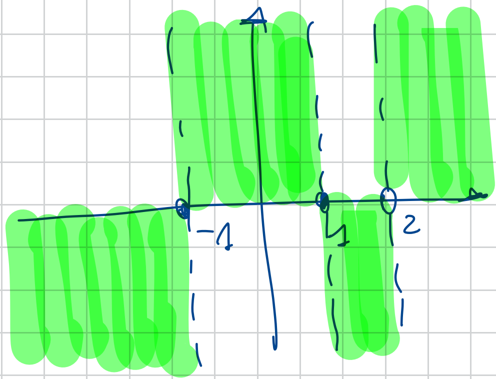


(3) SIGN OF $f(x)$

$$\frac{x^2-1}{x-2} \geq 0 \qquad \begin{cases} x^2-1 \geq 0 \\ x-2 > 0 \end{cases} \quad \begin{matrix} \to x \leq -1 ,\ x \geq 1 \\ \to x > 2 \end{matrix}$$

how combine?





1) ASYMPTOTES

$$\lim_{x\to+\infty} \frac{x^2-1}{x-2} = \lim_{x\to+\infty} \frac{\cancel{x^2}\left(1-\cancel{\frac{1}{x^2}}^{\,0}\right)}{\cancel{x}\left(1-\cancel{\frac{2}{x}}^{\,0}\right)} = +\infty \longrightarrow \text{no horiz.}$$

$$\lim_{x\to-\infty} \frac{x^2-1}{x-2} = -\infty \longrightarrow \text{no horiz.}$$

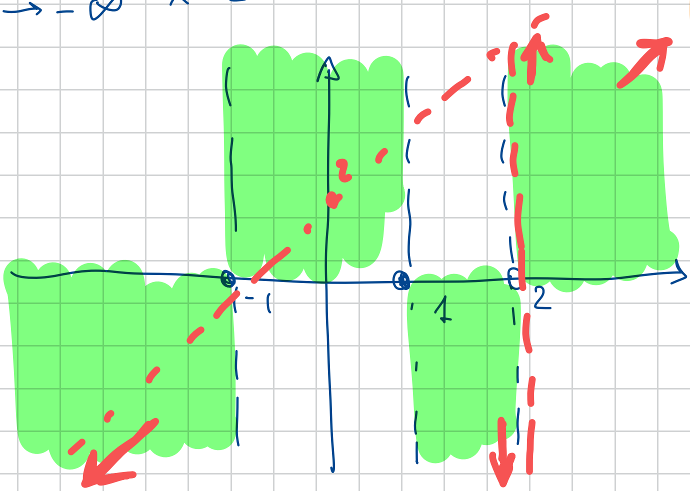
oblique?

$y = mx + q$, $m \to 1$, $q \to 2$

$$\lim_{x\to\pm\infty} \frac{x^2-1}{x-2}\cdot\frac{1}{x} = m = 1$$

$$\lim_{x\to\pm\infty} \frac{x^2-1}{x-2} - 1\cdot x = q = 2$$

$\exists$ oblique asymptote = $y = x+2$



$$\left.\begin{aligned}\lim_{x\to 2^+} \frac{x^2-1}{x-2} &= \frac{3}{0^+} = +\infty \\ \lim_{x\to 2^-} \frac{x^2-1}{x-2} &= \frac{3}{0^-} = -\infty\end{aligned}\right]\ \text{vert. asymp.}$$

$y = 2$

⑤ MAX & MIN

$$f'(x) = \frac{x^2 - 4x + 1}{(x-2)^2}$$

stationary points? $\quad f'(x) = 0 \quad \dfrac{x^2-4x+1}{(x-2)^2} = 0$

$x_1 = 2+\sqrt{3}$, $x_2 = 2-\sqrt{3}$

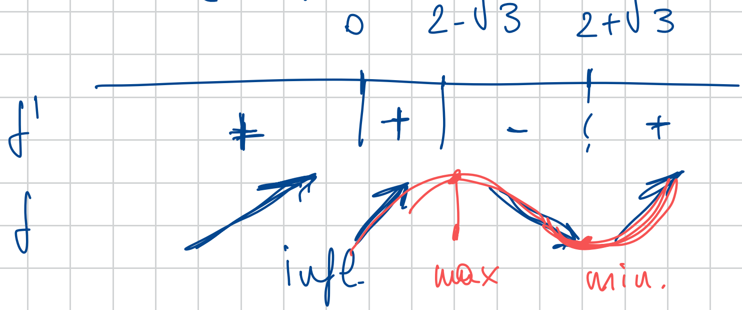
sign of f'

$f'(x) > 0$

$$\frac{x^2-4x+1}{(x-2)^2} > 0$$

$x^2 - 4x + 1 > 0 \rightarrow x < 2-\sqrt{3}$, $x > 2+\sqrt{3}$

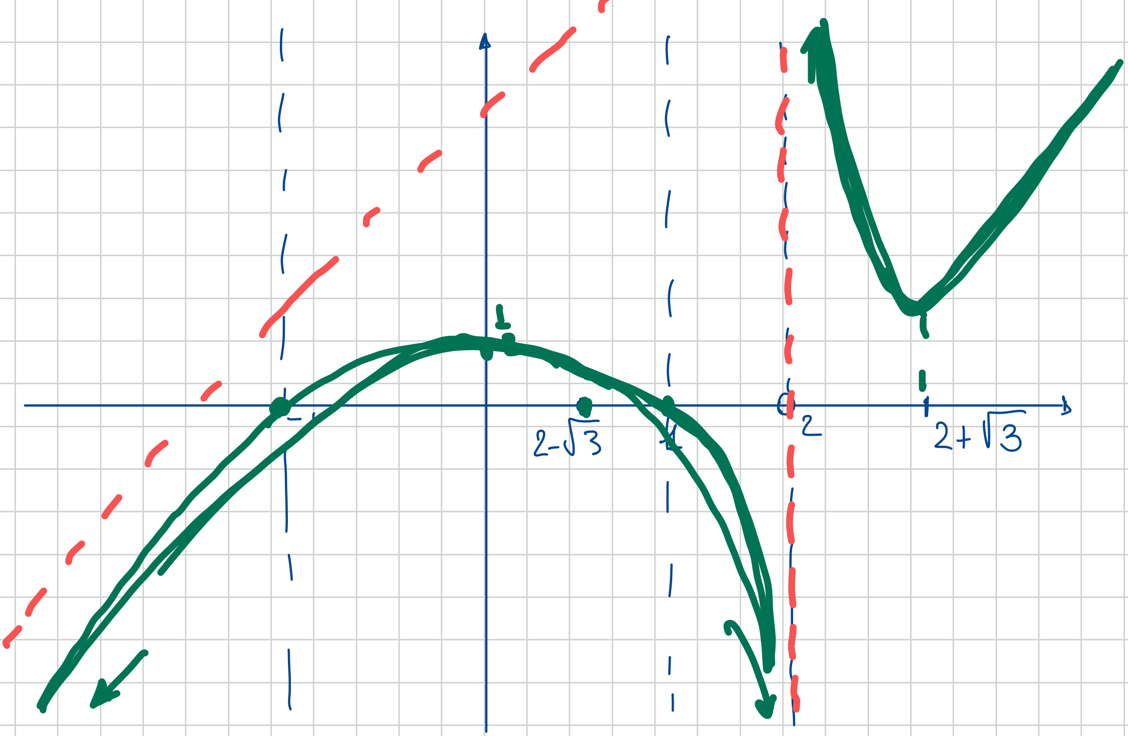
$(x-2)^2 > 0 \rightarrow \forall x \in \mathbb{R}$



⑥ CONCAVITY

$$f''(x) = \frac{6x-12}{(x-2)^4} \longrightarrow = 0 \quad x = 2 \longrightarrow \text{no inflect.}$$

$\hookrightarrow x = 2 \notin D_f$



$2-\sqrt{3}$
2
$2+\sqrt{3}$

$$f(x) = \frac{7}{x^2+1} - 3$$

DOMAIN $\longrightarrow$ $x^2+1 \neq 0$ $\quad x^2 \neq -1 \rightarrow \forall x \in \mathbb{R} \rightarrow D = \mathbb{R}$

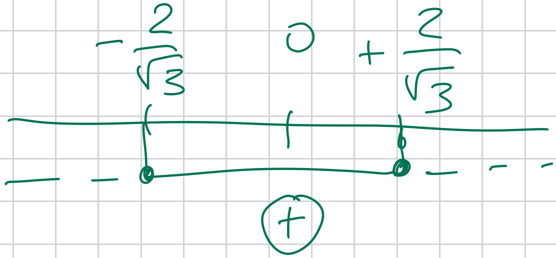
INTERS. $\rightarrow$ y $\begin{cases} x = 0 \\ y = \dfrac{7}{x^2+1} - 3 \end{cases} \longrightarrow y = \dfrac{7}{0+1} - 3 = 4$ $\quad A(0,4)$

x $\begin{cases} y = 0 \\ y = \dfrac{7}{x^2+1} - 3 \end{cases} \longrightarrow x = \pm \dfrac{2}{\sqrt{3}}$ $\quad B\left(-\dfrac{2}{\sqrt{3}}, 0\right)$ $\quad C\left(+\dfrac{2}{\sqrt{3}}, 0\right)$

SIGN $\rightarrow$ $f(x) \geq 0$

$$\frac{7}{x^2+1} - 3 \geq 0 \longrightarrow \frac{-3x^2+4}{x^2+1} \geq 0$$

$$\begin{cases} -3x^2+4 \geq 0 \longrightarrow 3x^2 - 4 \leq 0 \\ x^2+1 > 0 \longrightarrow \forall x \in \mathbb{R} \end{cases} \longrightarrow -\frac{2}{\sqrt{3}} \leq x \leq +\frac{2}{\sqrt{3}}$$



ASYMPTOTES

$$\lim_{x \to \pm\infty} \frac{4-3x^2}{x^2+1} = \frac{4-\infty}{+\infty+1} = \frac{\infty}{\infty} \text{ INDET. FORM.} = \frac{-3}{1} = -3$$

horiz. as.

$$y = -3$$

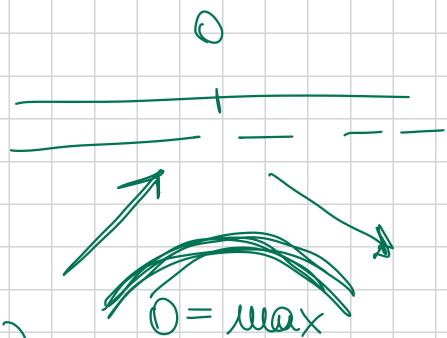
∄ vertic. as.

∄ oblique as.

MONOTONICITY

$$f'(x) = \frac{-14x}{(x^2+1)^2} \longrightarrow f'(x) > 0 \quad \frac{-14x}{(x^2+1)^2} > 0$$

$$\begin{cases} -14x > 0 \rightarrow x < 0 \\ (x^2+1)^2 > 0 \rightarrow \forall x \in \mathbb{R} \end{cases}$$

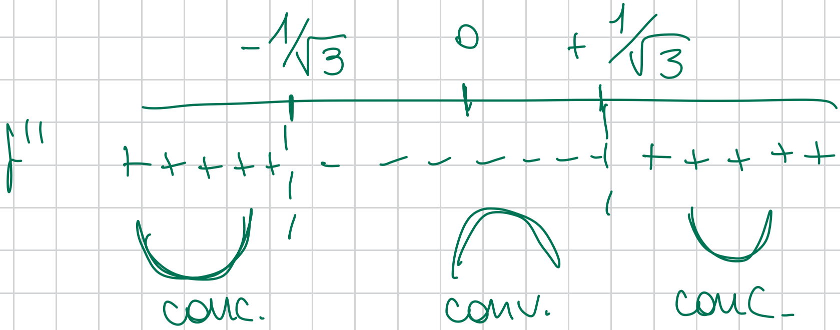


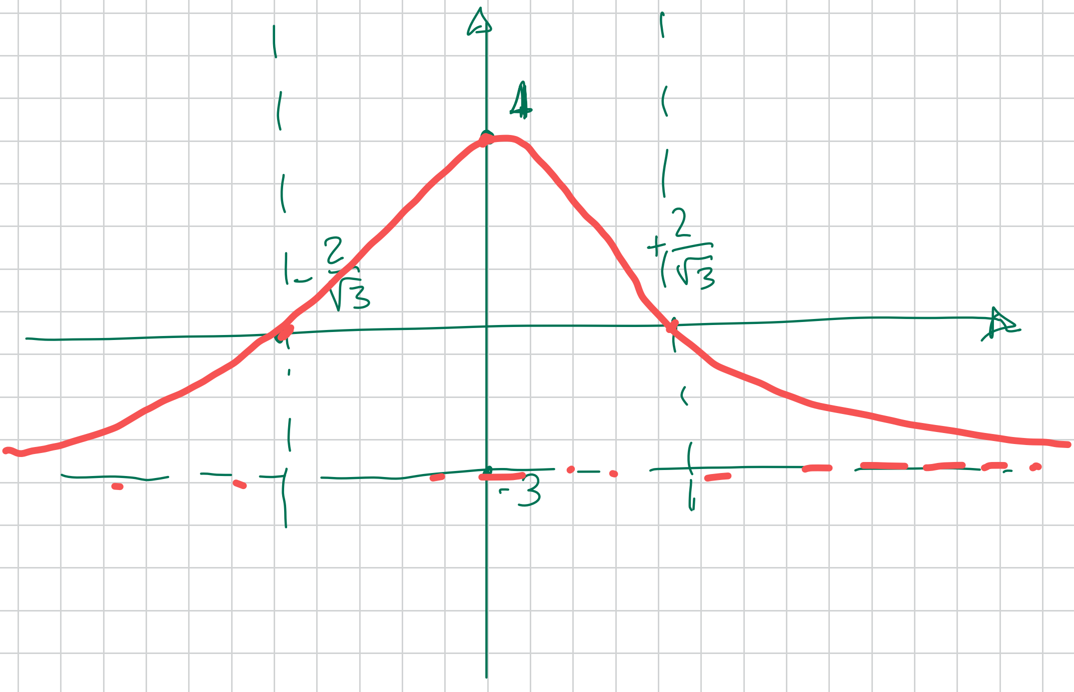
$$f(0) = 4 \rightarrow \max (0, 4)$$

CONCAVITY

$$f''(x) = \frac{-14(-3x^2+1)}{(x^2+1)^3} > 0 \qquad \begin{cases} -14-(3x^2+1) > 0 \\ (x^2+1)^3 > 0 \end{cases}$$

$$\begin{cases} 3x^2-1 < 0 \longrightarrow x < -\frac{1}{\sqrt{3}},\ x > \frac{1}{\sqrt{3}} \\ x^2+1 > 0 \longrightarrow \forall x \in \mathbb{R} \end{cases}$$


$-1/\sqrt{3}$
0
$+1/\sqrt{3}$
f''
+ + + + + + - - - - - - - - + + + + + +
conc.
conv.
conc.


4
$-\frac{2}{\sqrt{3}}$
$+\frac{2}{\sqrt{3}}$
-3

TRY ALONE:

$$f(x) = \frac{x^2+3}{x}$$

$$f(x) = \frac{x^2+2x-15}{x^2-9}$$

$$f(x) = \frac{1}{x^2+x-2}$$