- **INTEGRALS**

Integral

The integration procedure:
Determine the function given its derivative (the position of a body given its velocity, the size of a population given the growth rate, …)

- Integration also has the meaning of summation (area of a portion of the plane with curved boundaries)
- The two interpretations are closely related

The concept of the integral is linked to the solution of two classes of problems:

Indefinite Integral:

- Calculation of the analytic expression of a function from the derivative of the function itself

- **Definite Integral**:
  - calculation of the areas of figures bounded by curves
  - calculation of volumes
  - calculation of the work done by a force
  - …..

Integral calculus: area under the graph of a function

The problem of integration for real functions of a real variable →this problem is historically linked to the problem of measure (computing the area of figures with 'curvilinear' boundaries).

We will relate the problem of integration to the problem of the antiderivative, namely:

Given a function $f : I \to \mathbb{R}$, determine (all) functions $F : I \to \mathbb{R}$ such that

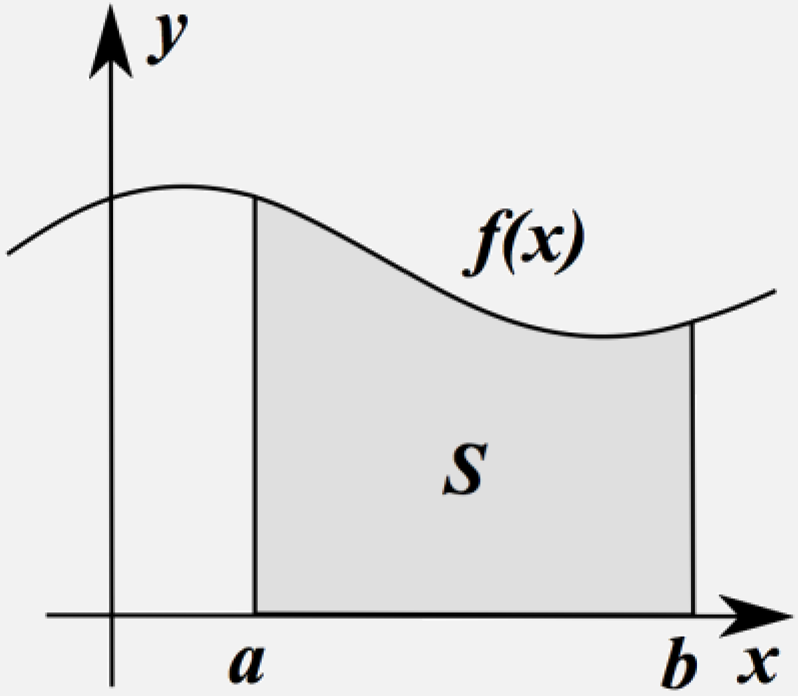
$$F'(x) = f(x) \quad \forall x \in I$$

F is called an antiderivative of F'

Motivations: computing an area

Let $f : a, b \to \mathbb{R}$ continuous and <u>positive</u>.
Compute the area A of the planar region (called the subgraph of f) contained between the graph of f and the x-axis.

This problem makes sense, because we assume $f \geq 0$.
Otherwise it would make no sense to speak of the subgraph of f.

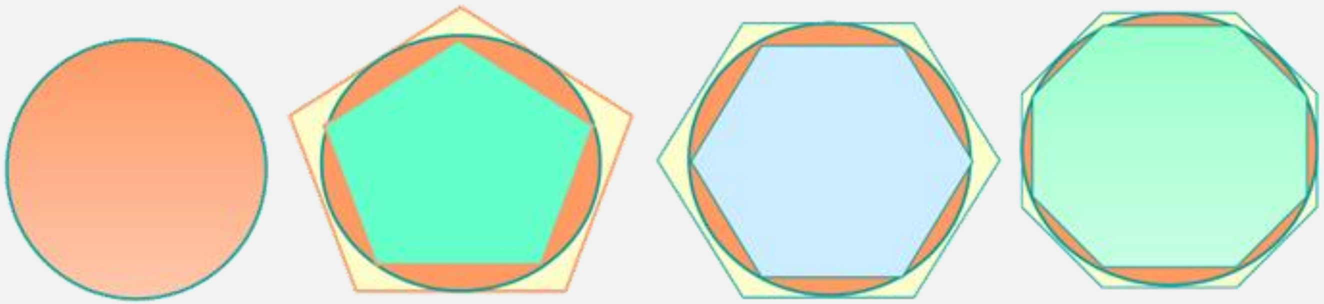


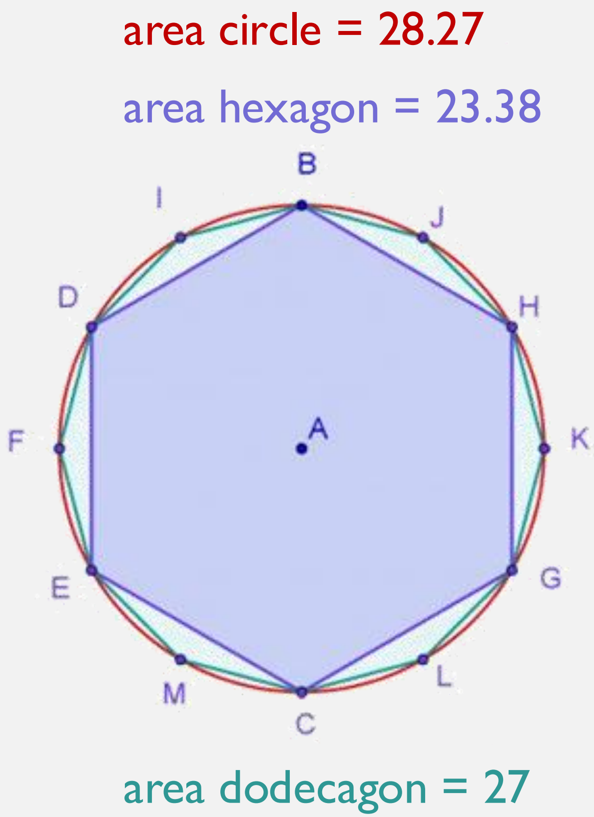
Example: area of the circle.

Archimede di Siracusa (287 a.C. – 212 a.C.): method of exhaustion for the area of the circle/area subtended by a branch of a parabola

It is possible to calculate the area through successive approximations by means of regular polygons inscribed in the circle and regular polygons circumscribed about the circle.

It is shown that the area of the circle is equal to the common limit, when the number of sides tends to ∞, to which the sequences formed by the areas of the polygons inscribed in and circumscribed about the circle tend.

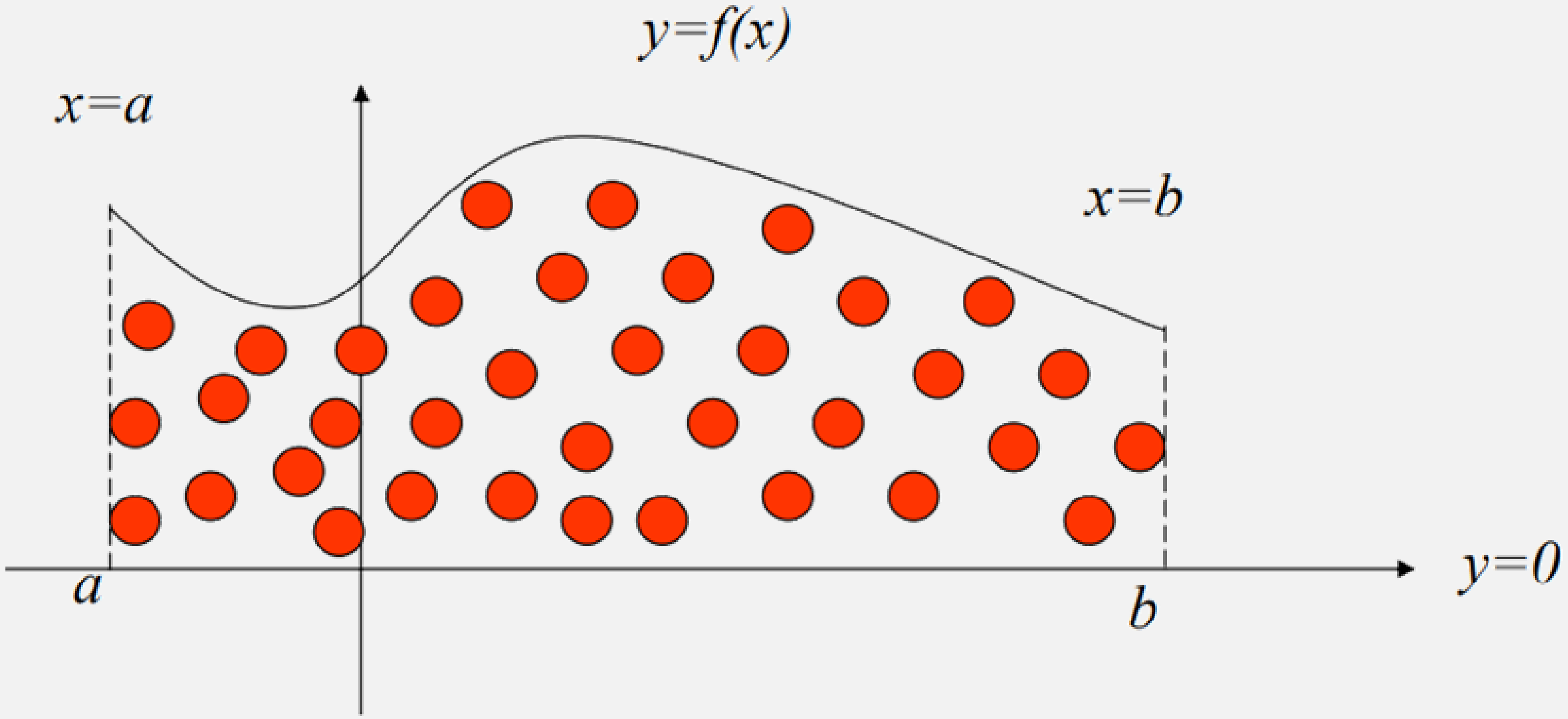




The concept of the integral: calculation of areas

Area of the rectangle relative to a function

Given a function f continuous on the interval $[a, b]$, we wish to compute the area of the plane region comprised between the x-axis, the two vertical lines with equation $x = a$ and $x = b$ and the graph of f. Such a plane region is called the rectangle relative to the function f.



Definite integral of a function: definition

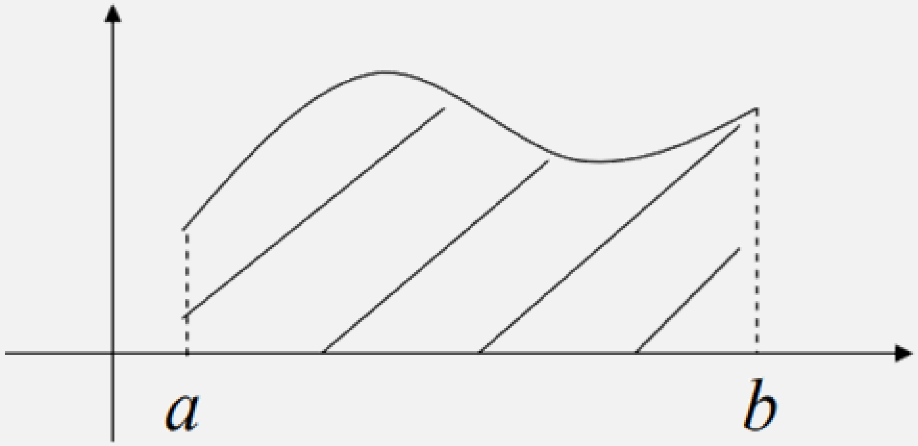
Given a function f continuous on the interval $[a, b]$, the **definite integral** of the function $f(x)$ in the interval $[a, b]$ is the measure of the area of the rectangular prism R associated with the function f and is denoted by the symbol:

$$\int_a^b f(x)dx = Area\ R$$

The problem is to determine the area of a region of the plane R which lies, in an orthogonal Cartesian coordinate system, above the x-axis and below the graph of a continuous and non-negative function $f(x)$ as the variable x varies over an interval $[a, b]$ closed and bounded.

At this regard, let $f(x)$ be a function defined and continuous on an interval $[a, b]$ that is closed and bounded, and let $f(x) \geq 0$ as the variable x varies in $[a, b]$.

Under these assumptions, we have:

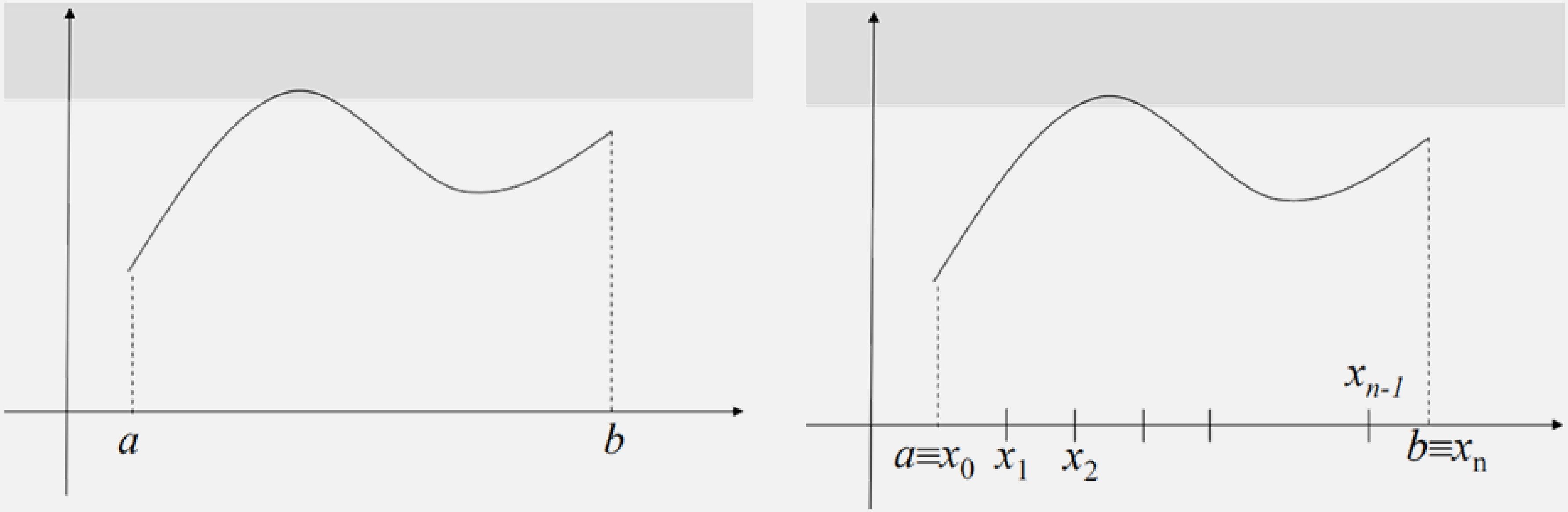


We define **rectanguloid** associated with the function f the region of the plane between the graph of $f \geq 0$ and the x-axis:

$$R = \{(x, y): a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$$

Always under this hypothesis, we subdivide the interval $[a, b]$ into a number n of equal parts by means of division points:

$$a \equiv x_0, x_1, x_2, \dots, x_{n-1}, x_n \equiv b$$



In this way, we subdivide the interval $[a, b]$ into n smaller subintervals, equal to each other:

$$[x_{i-1}, x_i], i = 1, \dots, n$$

each of width $h = \frac{b-a}{n}$

Since $f(x)$ is defined and continuous in $[a, b]$, closed and bounded interval, we have that $f(x)$ is defined and continuous in each of the partial intervals $[x_{i-1}, x_i]$

Applying Weierstrass' theorem in each partial interval

↓

A function $f(x)$ continuous in a closed and bounded interval $[a, b]$ has a minimum $m = f(x_1)$ and a maximum $M = f(x_2)$ such that

$$f(x_1) \leq f(x) \leq f(x_2), \qquad \forall x \in [a, b]$$

The function admits a minimum and a maximum, which are, in particular ,the points x_1 and x_2 of the interval $[a, b]$.

in each of the partial intervals $[x_{i-1}, x_i]$ the function f admits minimum m_i and maximum M_i, with $i = 1, \dots, n$

Starting from the minima m_i in each of the partial intervals, it is possible to build n rectangles whose base is always equal to the width h of each partial interval, and whose height is equal to m_i

It is hence possible to compute the area of each of these rectangles.
For example, the area of the first is:

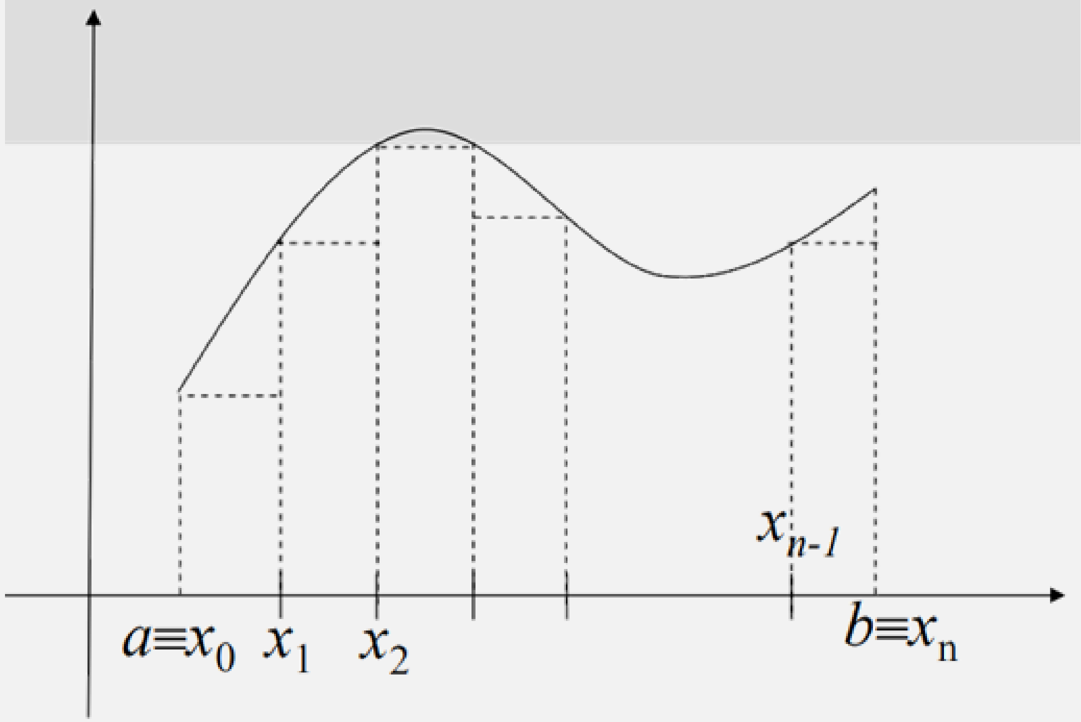
$$m_1(x_1 - x_0) = m_1 \frac{b-a}{n} = m_1 h$$

Similarly, the area of the second is:

$$m_2(x_2 - x_1) = m_2 \frac{b-a}{n} = m_2 h$$

And the area of the n-th is:

$$m_n(x_n - x_{n-1}) = m_n \frac{b-a}{n} = m_n h$$



Hence, the sum of the area of the n rectangles whose height is m_i is:

$$s_n = m_1 h + m_2 h + \cdots + m_n h$$

Starting from the maxima Mi in each of the partial intervals, one can construct n rectangles whose base is always equal to the width h of each partial interval, and whose height is equal to Mi

It is then possible to calculate the area of each of these rectangles.
For example, the area of the first is:

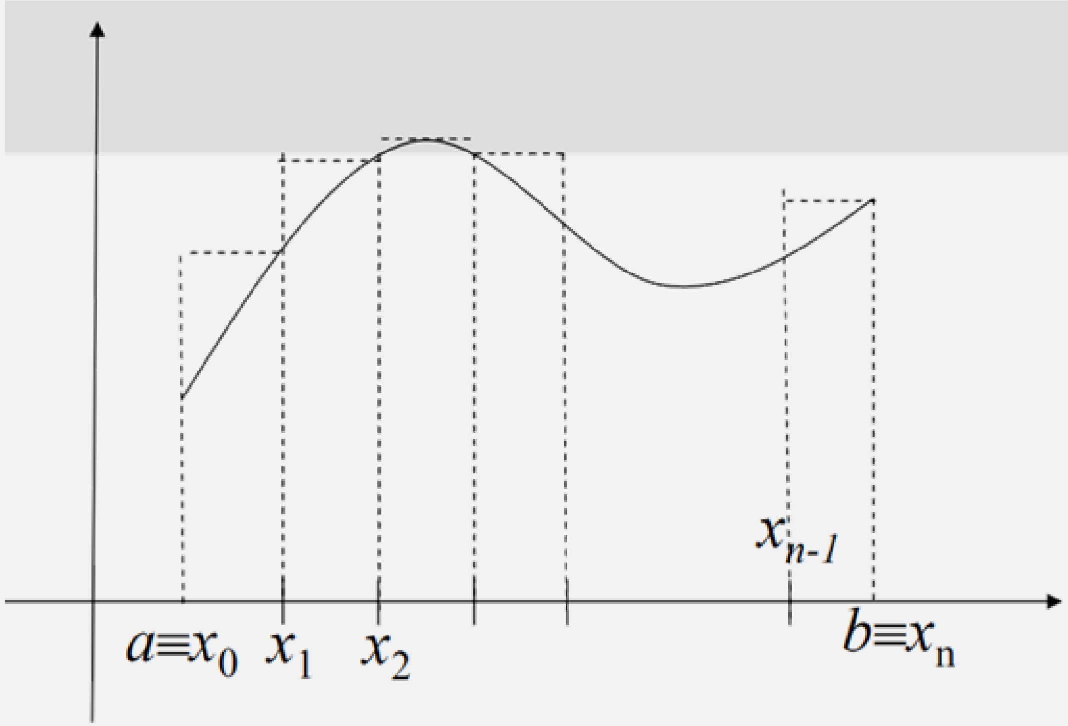
$$M1h$$

Analogously, the area of the *second* is:

$$M2h$$

And that of the n-th *is*:

$$Mn\mathrm{h}$$



Thus the sum of the areas of the n rectangles of height Mi is:

$$Sn = M1y + M2h + \cdots + Mn\mathrm{h}$$

It certainly holds that:

$$sn \leq Sn, \forall n \in \mathbb{N}$$

That is, whatever the subdivision of the interval $[a, b]$ into subintervals, the area of the multi-rectangle inscribed in the rectangleoid is always less than or equal to that of the multi-rectangle circumscribed about the rectangleoid.

Theorem.
If $f(x)$ is a function defined and continuous in $[a,b]$, closed and bounded interval, and if $f(x) \geq 0$ as the variable x changes on $[a,b]$,

Then, the sums s_n and S_n have finite limite as $n \to +\infty$ and in particular have the same limit, coincident with the area of the rectanguloid built from the function f:

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} S_n = Area\ R$$

In particular, the common value of the limit of the sums s_n and S_n is called **definite integral** of the function $f(x)$ extended to the interval $[a,b]$ and is indicated by:

$$\int_a^b f(x)dx = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} S_n$$

This simply corresponds to the area of the rectanguloid built from the function f (let us recall that $f(x) \geq 0$ as the variable x changes in $[a,b]$

The numbers a and b (endpoints of the function's domain) are defined as **integration limits** and, in particular:

- a and the lower limit of integration
- b is the upper limit of integration

The function $f(x)$ is defined as the **integrand**
The variable x is the **variable of integration**

Ultimately, by definition one has:

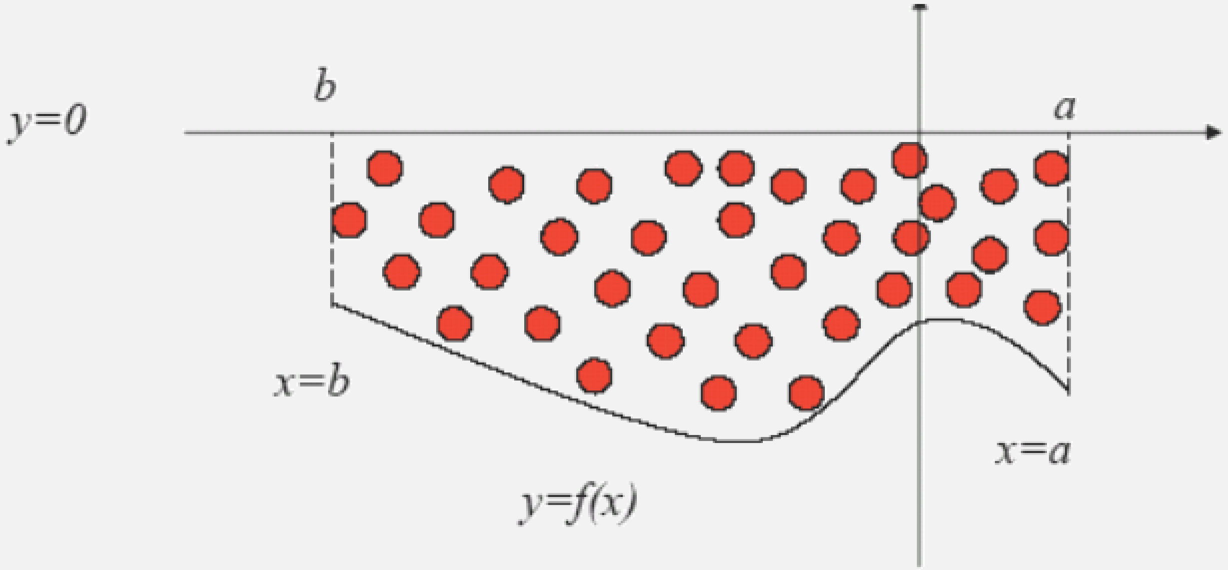
$$\int_a^b f(x)dx = \lim_{n \to +\infty} s_n = \lim_{n \to +\infty} S_n = Area\ R$$

Under the stated assumptions, the definite integral is a number greater than zero.

Observation 1.
The definition of the definite integral of a function f defined and continuous on an interval $[a, b]$, in the particular case in which $f(x) \geq 0$, has a geometric interpretation, since it coincides with the area of the curvilinear trapezoid associated with the function itself.

And what if $f(x)$ is not always greater than 0?

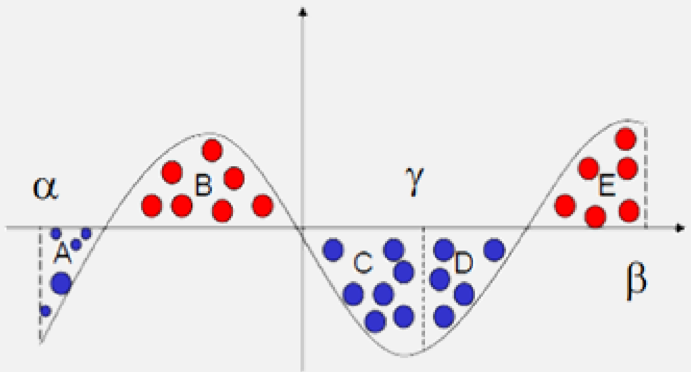
The sums s_n and S_n associated with a function $f(x)$, defined and continuous on an interval $[a, b]$, can also be constructed independently of the sign of the function $f(x)$ itself on the interval $[a, b]$



In particular, given $f(x)$ continuous on $[a, b]$ (not necessarily positive), the definite integral

$$\int_a^b f(x)dx$$

is interpreted as the sum of the areas of the regions that the graph $f(x)$ identifies together with the horizontal axis and the lines $x = a$ and $x = b$



$$\int_\alpha^\gamma f(x)dx = -A + B - C$$

$$\int_\alpha^\beta f(x)dx = -A + B - C - D + E$$

$$\int_\alpha^\gamma f(x)dx = \int_\alpha^\beta f(x)dx + \int_\beta^\gamma f(x)dx$$

Properties of the definite integrals

Let $f, g : [a, b] \to \mathbb{R}$ integrable. Then, $\forall \lambda \in \mathbb{R}$ the functions

$$f + g, \quad \lambda f, \quad |f| \text{ are integrable}$$

$$\forall [c, d] \subset [a, b], \quad f_{[c,d]} \text{ is integrable}$$

Moreover, the following properties hold:

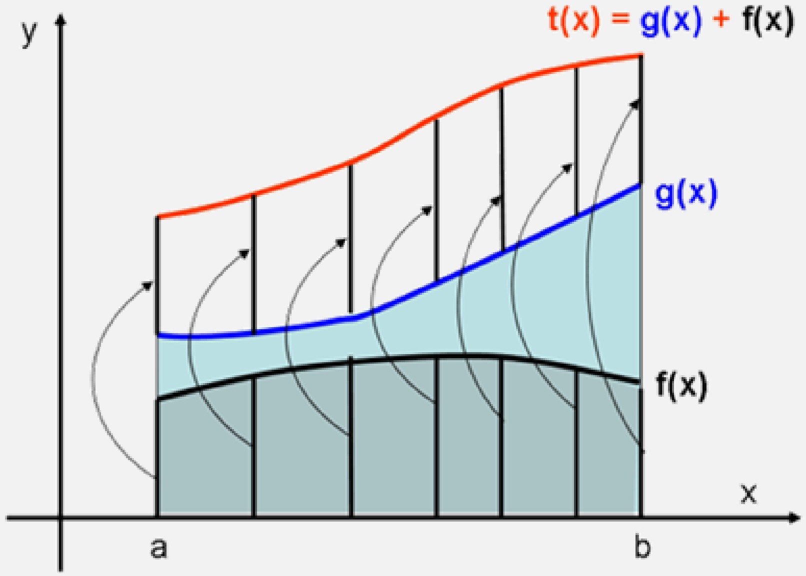
- **Linearity** $\forall \alpha \in \mathbb{R}$

$$\int_a^b \big(\alpha f(x)\big) dx = \alpha \int_a^b f(x) dx$$

$$\Longrightarrow \int_a^b -f(x) dx = -\int_a^b f(x) dx$$

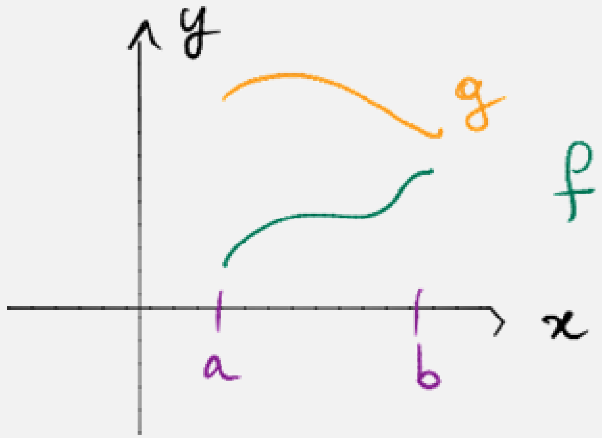
- Integral of the sum:

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$



- Comparison property: if $f(x) \leq g(x), \forall x \in [a, b]$

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

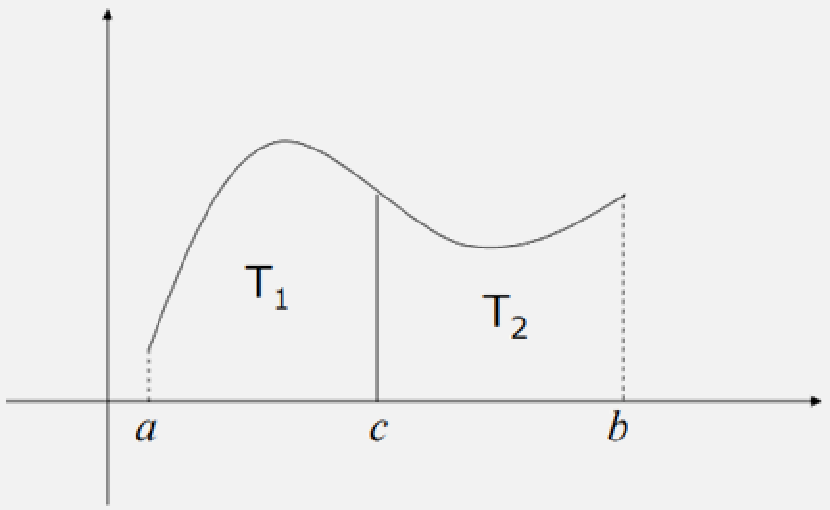


- **Additive property:** $\forall c \in (a,b) : a < c < b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Clear geometric meaning with positive functions:

$$Area\ R = Area\ T_1 + Area\ T_2$$



First Fundamental Theorem of Integral Calculus

Let $f(x)$ be a continuous and positive function on $[a, b]$.
Fixed x in $[a, b]$, we define:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

In other words, the integral provides us with a way to construct a function with a prescribed derivative. In other words, the function

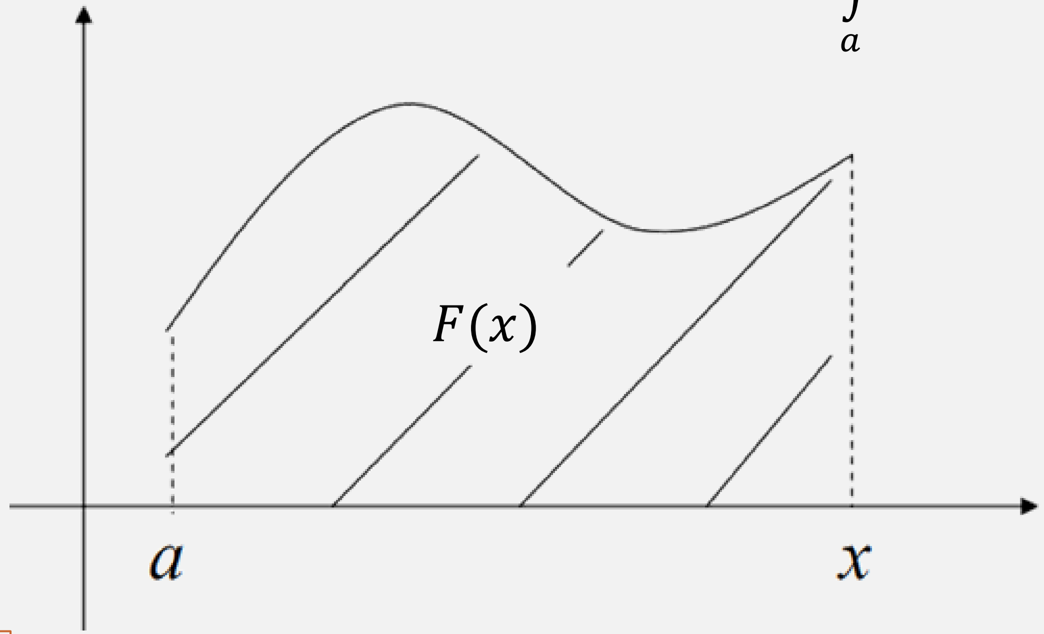
$$F(x) = \int_a^x f(t)dt$$

is a solution of the equation

$$\frac{dF}{dx} = f$$

Recalling that two functions with the same derivative differ only by an additive constant, we obtain that:

$$F(x) = \int_a^x f(t)dt$$



The solutions of $\frac{dF}{dx} = f$ are all and only of *the* form:

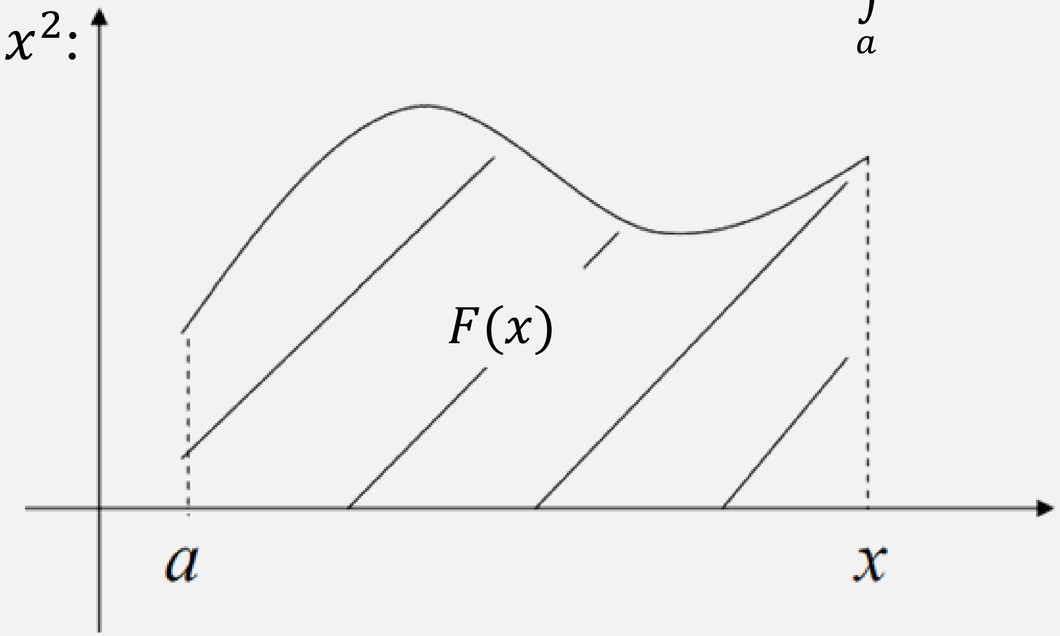
$$F(x) = \int_a^x f(t)dt + c$$

with $c \in \mathbb{R}$

Practical example:

In the interval $[0, x]$, the integral of the function $2t$ is x^2:

$$F(x) = \int_a^x f(t)\,dt$$



$$F(x) = \int_0^x 2t\,dt = x^2$$

The **integrand function** $f(x)$ is $f(x) = 2x$

The **integral function** $F(x)$ is $F(x) = x^2$

The derivative of the integral function is

$$F'(x) = \frac{d}{dx}x^2 = 2x$$

Hence, the derivative of the integral function $F'(x)$ is equal to the integrand function $f(t)$ calculated for $t = x$:

$$F'(x) = f(x)$$

Second Fundamental Theorem of Integral Calculus (Torricelli-Barrow Theorem)

Let $f(x) : [x_0, x_1] \to \mathbb{R}$ be a differentiable function; for every $x \in [x_0, x_1]$ one has:

$$\int_{x_0}^{x} \frac{d}{dt} f(t)dt = f(x) - f(x_0)$$

One may use the following abbreviated form to denote this difference:

$$f(x) - f(x_0) = f(t)|_{x_0}^{x}$$

The Second Fundamental Theorem of Integral Calculus provides a first method to calculate some forms of integrals. Let us start from the definition:

$$\int_a^b f(t)dt = F(b) - F(a)$$

From the formula $\frac{d}{dx}x^{n+1} = (n+1)x^n$ we deduce that $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = x^n$.

Hence:

$$\int_a^b x^n dx = \frac{1}{n+1}x^{n+1}|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}$$

In this way, we can calculate the integral of whatever polynomial:

$$\int_1^2 (6x^2 + 2x - 3)dx = 6\int_1^2 x^2 dx + 2\int_1^2 xdx - 3\int_1^2 dx =$$

$$= 6\frac{1}{3}x^3|_1^2 + 2\frac{1}{2}x^2|_1^2 - 3(2-1) = 14$$

Indefinite integral – Antiderivative function

If f is the derivative of the function F, it is said that F is an **indefinite integral** (or an **antiderivative**) of f and it is written:

$$\int f(t)dt = F(t) + c$$

The reason for this notation is that two indefinite integrals of f differ by an additive constant (since they have the same derivative): therefore, as the constant $c \in \mathbb{R}$ varies, the right-hand side of the equation describes all possible indefinite integrals (or all antiderivatives) of f.

Moreover:

$$\int f(t)dt = F(t) + c \Leftrightarrow \int_a^b f(t)dt = F(t)|_a^b$$

In the definition provided, the limits of integration do not appear: the formula is valid on every interval $[a, b]$ where f is the derivative of F.

Antiderivative function

A function $F(x)$ defined and differentiable on $[a,b]$, is called **antiderivative** of $f(x)$, defined and continuous on $[a,b]$, if it holds that:

$$F'(x) = f(x), \qquad \forall x \in [a,b]$$

<u>Observation</u>.
If $F(x)$ is an antiderivative of the function $f(x)$ (hence $F'(x) = f(x)$), then $F(x) + c$ is still an antiderivative of $f(x), \forall c \in \mathbb{R}$ and vice versa (in fact: $(F(x)+c)' = f(x), \forall c \in \mathbb{R}$)

⇓

if $F(x)$ is an antiderivative of f, all the antiderivatives of f are obtained by adding a constant to F

Characterization of the antiderivatives of a function

If $F(x)$ and $G(x)$ are two antiderivatives of the same function f on an interval

Then $F(x)$ and $G(x)$ differ by a constant, that is:

$$F(x) = G(x) + c, \qquad \forall c \in \mathbb{R}$$

Theorem: by adding a constant to an antiderivative, one still obtains an antiderivative

On this basis, it follows that if a function f admits an antiderivative, then it has infinitely many

If $F(x)$ and $G(x)$ are by hypothesis two antiderivatives of the same function f , this means that by hypothesis it holds that:

$$F'(x) = f(x), \forall x \in [a, b]$$

$$G'(x) = f(x), \forall x \in [a, b]$$

from this, it immediately follows that the derivative of the difference between F and G is:

$$D\big(F(x) - G(x)\big) = F' - G' = f(x) - f(x) = 0, \forall x \in [a, b]$$

$$\Rightarrow \big(F(x) - G(x)\big) = \text{constant}$$

Example. Compute the antiderivative of the following functions.

$f(x) = x$
$F(x) = \frac{x^2}{2}$, in fact $F'(x) = \frac{1}{2}2x = x = f(x)$

Moreover, also $F(x) = \frac{x^2}{2} + 1$ is an antiderivative of the function $f(x) = x$

Given the theorem on the sum of a constant and an antiderivative function, we can write:

$$F(x) = \frac{x^2}{2} + c$$

$f(x) = e^x$
$F(x) = e^x + c$ → in fact $F'(x) = e^x = f(x)$

$f(x) = \frac{1}{x}$
$F(x) = \ln x + c$ → in fact $F'(x) = \frac{1}{x} = f(x)$

$f(x) = x^3$

$F(x) = \frac{1}{4}x^4 + c$, in fact $F'(x) = \frac{1}{4}4x^{4-1} = x^3 = f(x)$

$f(x) = \frac{1}{x^2}$

$F(x) = -\frac{1}{x} + c$, in fact $F'(x) = +x^{-1-1} = x^{-2} = f(x)$

$$F(x) = G(x) + c, \qquad \forall c \in \mathbb{R}$$

Theorem: by adding a constant to an antiderivative, one still obtains an antiderivative

On this basis, it follows that if a function f admits an antiderivative, then it has infinitely many

The set of functions $Gx + c, \forall c \in \mathbb{R}$ represents exactly all the functions whose derivative is $f(x)$ and is called the indefinite integral of f, which is denoted by the following symbol:

$$\int f(x)\,dx$$

By definition:

$$\int f(x)\,dx = F(x) + c \iff F'(x) = f(x)$$

$$\int f(x)\,dx = F(x) + c \;\Leftrightarrow F'(x) = f(x)$$

The indefinite integrals have properties that derive from the fact that the integral is a linear operator, meaning $D[c_1 \cdot f_1(x) + c_2 \cdot f_2(x)] = c_1 \cdot f_1'(x) + c_2 \cdot f_2'(x)$.

This is reflected on the following properties of indefinite integrals:

- The integral of the product between a constant and a function is equal to the product between the constant and the integral of the function. A multiplicative constant $k \in \mathbb{R}$ can hence be brought inside or outside the integral sign:

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx$$

- The integral of the sum of two or more functions is equal to the sum of the integral calculated on each function:

$$\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$$

- Combining the two properties, the indefinite integral results to be a linear operator → the integral of a linear combination of functions is equal to the linear combination of their integrals: for each k_1 and k_2 being real constants and for each function f_1 and f_2 it holds:

$$\int [k_1 \cdot f_1(x) + k_2 \cdot f_2(x)]dx = k_1 \int f_1(x)dx + k_2 \int f_2(x)dx$$

Example: compute the integral of a polynomial

$$f(x) = 10x^4 + 4x^3 + 5x^2 + 3$$

$$\int f(x)dx = \int [10x^4 + 4x^3 + 5x^2 + 3]dx = 10\int x^4 dx + 4\int x^3 dx + 5\int x^2 dx + 3\int dx$$

$10 \int x^4 dx = 10 \cdot \left[\frac{x^5}{5} + c_1\right] = 2x^5 + c_1$

$4 \int x^3 dx = 4 \cdot \left[\frac{x^4}{4} + c_2\right] = x^4 + c_2$

$5 \int x^2 dx = 5 \cdot \left[\frac{x^3}{3} + c_3\right] = \frac{5}{3}x^3 + c_3$

$3 \int dx = 3 \int x^0 dx = 3 \cdot \left[\frac{x^1}{1} + c_4\right] = 3x + c_4$

By adding the terms (and indicating by a generic c all the constants):

$$\int f(x)dx = \int [10x^4 + 4x^3 + 5x^2 + 3]dx = 2x^5 + x^4 + \frac{5}{3}x^3 + 3x + c$$

The indefinite integral of a function $f(x)$ consists of all its antiderivatives, that is, all those functions whose derivatives yield precisely f.

Summary table of the most common antiderivatives:

1. Antiderivatives of functions: constant, power, root
2. Antiderivatives of trigonometric functions
3. Antiderivatives of exponential and logarithmic functions

Constant functions, powers (with natural or real exponent) and roots

Function $f(x)$	**Indefinite integral $\int f(x)dx$**
k (constant function)	$\int dx = \int kdx = kx + c$
x	$\int xdx = \frac{1}{2}x^2 + c$
x^α, with $\alpha \in \mathbb{R}, \alpha \neq -1$	$\int x^\alpha dx = \frac{1}{\alpha + 1}x^{\alpha+1} + c$
$\frac{1}{x} = x^{-1}$	$\int \frac{1}{x}dx = \ln\|x\| + c$

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Trigonometric functions

Function $f(x)$	**Indefinite integral $\int f(x)dx$**
$\sin x$	$\int \sin x \, dx = -\cos x + c$
$\cos x$	$\int \cos x \, dx = \sin x + c$
$\frac{1}{\cos^2 x}$	$\int \frac{1}{\cos^2 x} dx = \tan x + c$
$\frac{1}{\sin^2 x}$	$\int \frac{1}{\sin^2 x} = \cot x + c$
$\tan x$	$\int \tan x \, dx = -\ln\lvert\cos x\rvert + c$
$\cot x$	$\int \cot x \, dx = -\ln\lvert\sin x\rvert + c$

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1. Antiderivatives of functions: constant, power, root
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Exponential and logarithmic functions

Function $f(x)$	**Indefinite integral $\int f(x)dx$**
e^x	$\int e^x dx = e^x + c$
a^x, with $a \in \mathbb{R}, a \neq 1$	$\int a^x dx = \frac{1}{\ln a} a^x + c = a^x \log_a e + c$