APPLIED PHYSICS AND BIOSTATISTICS

MODULE MATH1 — Mathematical Toolkit

Numbers, Intervals, Equations and Inequalities

Prof. Valentina Notarstefano | Università degli Studi di Teramo

Why This Matters in the Lab

Most biomedical formulas are algebra you rearrange — e.g. solve Beer–Lambert $A = \varepsilon \cdot b \cdot c$ for the concentration c.

Reference ranges are intervals and inequalities — e.g. fasting glucose $70 \leq g \leq 99$ mg/dL.

Quadratic equations arise in equilibria and kinetics; logarithms turn exponential processes (pH, dose) into straight lines.

This unit recalls only the tools we will reuse throughout the course.

Number Sets

$\mathbb{N} = \{0,1,2,3,\ldots\}$ the set of natural numbers

$\mathbb{N}^* = \{1,2,3,\ldots\}$ the set of natural numbers different from zero

$\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$ the set of integers

$\mathbb{Q} = \{\frac{n}{m} : n, m \in \mathbb{Z}, m \neq 0\}$ the set of rational numbers (a rational number, written in its decimal form, after the comma shows a finite decimal expansion, or an infinite periodic one)

$\mathbb{R}$ the set of real numbers→ all the numbers that, written in their decimal form, show an infinite decimal expansion, also non-periodic

Number Sets

Among numerical sets, the following proper inclusions hold:

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$$

These inclusions are strict, because:

- there exist integers that are not natural numbers (negative integers);
- there exist rational numbers that are not integers (proper fractions);
- there exist real numbers that are not rational (irrational numbers).

In $\mathbb{N}$ there is neither an additive inverse nor a multiplicative inverse for any number → in $\mathbb{N}$ one can perform addition and multiplication, but in general it is not possible to perform the inverse operations of subtraction and division.

With the exception of 1, in $\mathbb{Z}$ no number has a multiplicative inverse → in $\mathbb{Z}$ subtraction is generally possible, whereas division is not.

Intervals

An empty I of the real number line is called an interval if $\forall x, y \in I$ every point lying between x and y also belongs to I

$A = \{x \in \mathbb{R} : x \geq 6\}$ → is an interval

$B = \{x \in \mathbb{R} : x \neq 0\}$ → is not an interval

Among the bounded intervals, for $a, b \in \mathbb{R} : a < b$, we have:

- Closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- Right-open interval $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- Left-open interval $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- Open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

All these intervals have lower endpoint a and upper endpoint b.

Intervals

Among unbounded intervals, we have:

- Closed interval unbounded above $[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}$
- Open interval unbounded above $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$
- Closed interval unbounded below $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$
- Open interval unbounded below $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$

Example — reference ranges are intervals

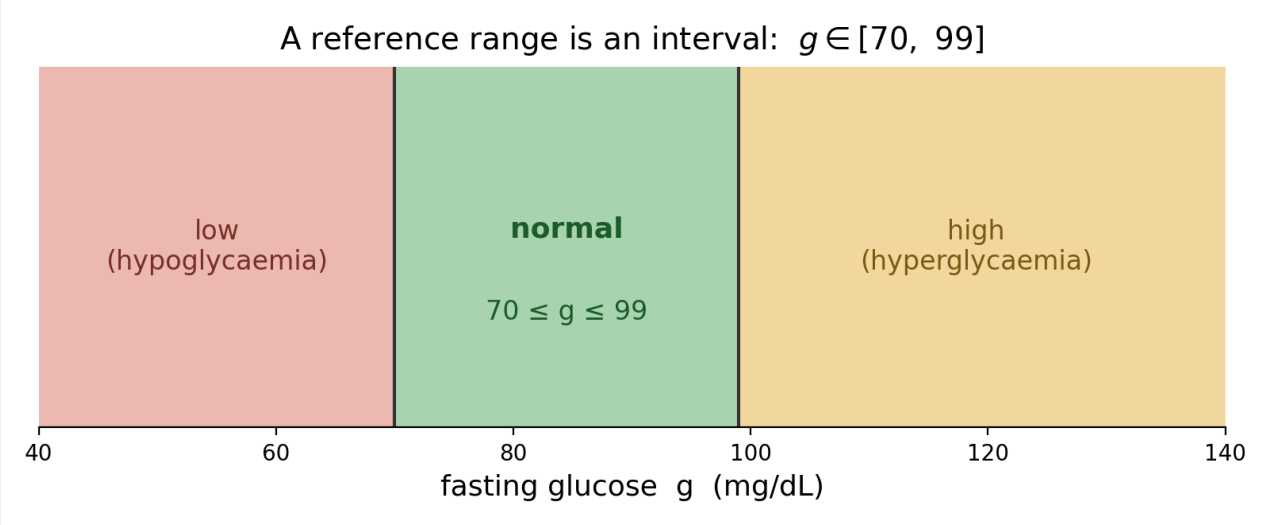
A clinical result is judged against a reference range — an interval of 'normal' values.

Values inside the interval are normal; outside it, they flag a possible condition.

For fasting glucose the normal interval is:

$70 \le g \le 99$ mg/dL $\quad\leftrightarrow\quad g \in [70, 99]$

Intervals and inequalities are exactly how every lab report is read.



Equations

Solving an equation means finding the unknown value, or values, for which the two algebraic expressions yield the same result.

A set of values that, when substituted for the unknowns, makes an equation true is called the solution set or the set of roots.

Therefore, solving an equation means explicitly determining the set of all its solutions.

Example.

Solving the equation

$$5x + 3 = 2x + 4$$

means finding the value of x for which

The left-hand side $5x + 3$ and the right-hand side $2x + 4$

Have the same value

Linear Algebraic Equations

A first-degree (or linear) algebraic equation is an equation in which the highest degree of the unknown is one and which, after suitable simplifications, can be written in the standard form:

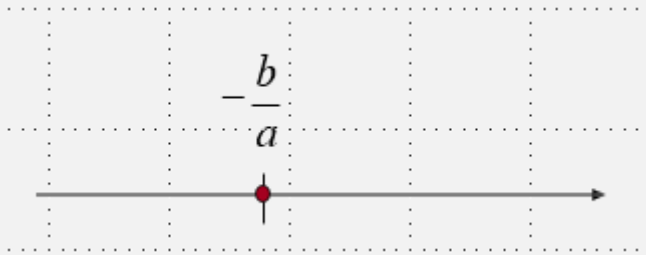
$$ax + b = 0$$

where $a \neq 0$ and $a, b \in \mathbb{R}$.

Solving for x gives:

$$x = -\frac{b}{a}$$

UNIQUE SOLUTION!



Special Cases

Given the equation:

$$ax + b = 0 \text{ with } a, b \in \mathbb{R}$$

If $a = 0$, one can distinguish two cases:

- $0 \cdot x + b = 0 \quad con\ b \neq 0$ → inconsistent equation (no solution)
- $0 \cdot x + b = 0 \quad con\ b = 0$ → indeterminate equation (infinitely many solutions)

Example. solve the following equation: $x + 6 = 21$

Adding to both sides number -6, the root set does not change:

$$x + 6 - 6 = 21 - 6$$

The solution is $x = 15$



Second-Degree Algebraic Equations

A second-degree (or quadratic) equation is an algebraic equation in one unknown x, with maximum degree equal to 2, and can be written in the standard form:

$$ax^2 + bx + c = 0,$$
$$a \neq 0, a, b, c \in \mathbb{R}$$

Example

The equation $3x^2 + 2x - 5 = 0$ is a second-degree algebraic equation written in its standard form with $a = 3, b = 2, c = -5$

Discriminant

The discriminant of a quadratic equation

$$ax^2 + bx + c = 0$$

Is defined as:

$$\Delta = b^2 - 4ac$$

Example

For the equation

$$3x^2 + 2x - 5 = 0$$

The discriminant is:

$$\Delta = b^2 - 4ac = 2^2 - 4(3)(-5) = 4 + 60 = 64$$

Nature of the Solutions

In contrast to first-degree equations, which always admit a unique solution, a second-degree equation may admit either no real solutions or two real solutions, which may coincide.

The existence and the qualitative nature of the real solutions are determined by the sign of the discriminant

$$\Delta = b^2 - 4ac$$

- if $\Delta = b^2 - 4ac > 0$

the equation admits two distinct real solutions, denoted by x_1 and x_2

→ And the following factorization holds:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

- if $\Delta = b^2 - 4ac = 0$

the equation admits two coincident real solutions, denoted by x_0

→ And the following factorization holds:

$$ax^2 + bx + c = a(x - x_0)^2$$

- if $\Delta = b^2 - 4ac < 0$

the equation admits no real solutions

Nature of the Solutions

- if $\Delta = b^2 - 4ac > 0$

the equation admits two distinct real solutions

- if $\Delta = b^2 - 4ac = 0$

the equation admits two coincident real solutions

if $\Delta = b^2 - 4ac \geq 0$ the two solutions are given by the formula:

$$x_{1,2} = -\frac{b \pm \sqrt{\Delta}}{2a}$$

Example. Solve the following equation: $x^2 + x - 2 = 0$

We compute the value of the discriminant:

$$\Delta = b^2 - 4ac = 1 + 8 = 9 > 0$$

Therefore, the equation admits two distinct real solutions.
We now determine them:

$$x_{1,2} = -\frac{1 \pm \sqrt{9}}{2 \cdot 1} = \begin{cases} x_1 = -\frac{4}{2} = -2 \\ x_2 = \frac{2}{2} = 1 \end{cases}$$

Absolute Value Equations

Definition

The absolute value of a real number x is defined as:

- x, if x is greater or equal to zero
- $-x$, if x is less than zero

$$|x| \rightarrow \begin{cases} x \geq 0 \\ x \end{cases} \vee \begin{cases} x < 0 \\ -x \end{cases}$$

Inequalities

In mathematics, an inequality is a relation of order between two expressions containing unknowns.

Solving an inequality means determining the set of values which, when assigned to the unknowns, make the inequality true.

Typically, the solutions of an inequality consist of one or more numerical sets, called intervals.

Inequalities

EQUIVALENCE PRINCIPLES

The solution set of an inequality does not change if:

- the same quantity (number or unknown) is added to both sides of the inequality;

consequently, the same term can be eliminated from both sides or moved from one side to the other by changing its sign;

- both sides of the inequality are multiplied or divided by the same quantity, provided that the direction of the inequality is reversed when this quantity is negative;

consequently, all terms on both sides may have their sign changed, provided that the direction of the inequality is also reversed.

First-Degree Algebraic Inequalities

An inequality is said to be first-degree (or linear) if, after suitable simplifications, it can be written in the form:

$$ax + b \geq 0 \qquad \text{or } > 0, \leq 0, < 0$$

where $a \neq 0$ and $\mathrm{a}, b \in \mathbb{R}$.

Recalling that the inequalities

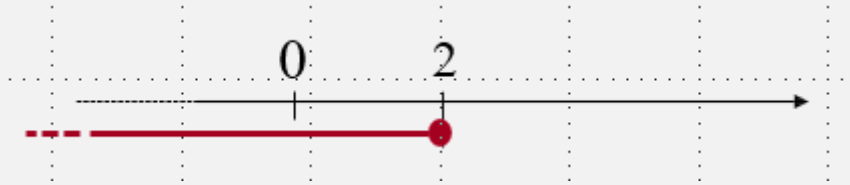
$$ax + b > 0 \text{ and } -ax - b < 0 \text{ with } a \neq 0$$

are equivalent, the resolution of first-degree inequalities can always be reduced to the case in which the coefficient a is positive.

Example. Solve the following inequality: $5x - 10 \leq 0$

Solution:

$$5x \leq 10 \rightarrow x \leq \frac{10}{5} \rightarrow x \leq 2$$



Second-Degree Algebraic Inequalities

A second-degree (or quadratic) inequality is an algebraic inequality in one unknown x, whose maximum degree is equal to 2 and which can be reduced to the form:

$$ax^2 + bx + c > 0$$

$$(ax^2 + bx + c < 0)$$

$$a \neq 0, a, b, c \in \mathbb{R}$$

Second-Degree Algebraic Inequalities – Solutions

$$ax^2 + bx + c > 0 \text{ or } (ax^2 + bx + c < 0)$$

$$a \neq 0, a, b, c \in \mathbb{R}$$

- Step 1: Verify the sign of the coefficient a. If $a < 0$, change the sign of all terms and reverse the direction of the inequality.

- Step 2: Determine the real roots x_1 and x_2 (if they exist) of the associated quadratic equation:

$$x_{1,2} = -\frac{b \pm \sqrt{\Delta}}{2a} \text{ with } \Delta = b^2 - 4ac$$

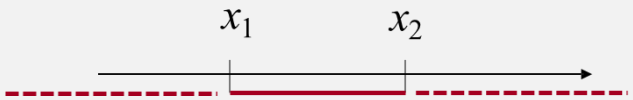
Second-Degree Algebraic Inequalities – Solutions

CASE I : $\Delta > 0, a > 0$ → Two distinct real solutions

- $ax^2 + bx + c > 0 \Leftrightarrow x < x_1,\ x > x_2$



- $ax^2 + bx + c < 0 \Leftrightarrow x_1 < x < x_2$



CASE I : $\Delta > 0, a > 0$ → Two distinct real solutions

- $ax^2 + bx + c \geq 0 \Leftrightarrow x \leq x_1,\ x \geq x_2$



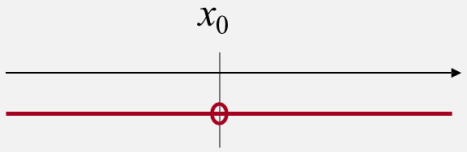
- $ax^2 + bx + c \leq 0 \Leftrightarrow x_1 \leq x \leq x_2$



Second-Degree Algebraic Inequalities – Solutions

CASE II : $\Delta = 0, a > 0$ → Two coincident real solutions

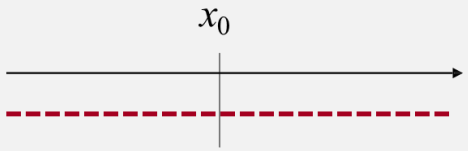
- $ax^2 + bx + c > 0 \Rightarrow \forall x \in R, x \neq x_0$



if the trinomial admits two coincident solutions equal to x_0 one can write:

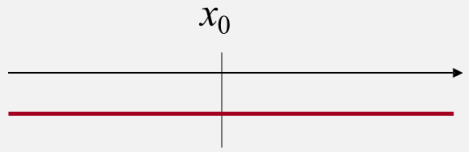
$$ax^2 + bx + c = a(x - x_0)^2$$

- $ax^2 + bx + c < 0 \Rightarrow \nexists x \in R$



CASE II : $\Delta = 0, a > 0$ → Two coincident real solutions

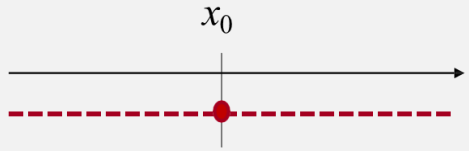
- $ax^2 + bx + c \geq 0 \Rightarrow \forall x \in R$



if the trinomial admits two coincident solutions equal to x_0 one can write:

$$ax^2 + bx + c = a(x - x_0)^2$$

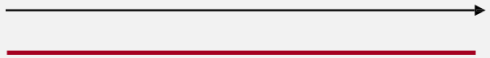
- $ax^2 + bx + c \leq 0 \Rightarrow x = x_0$



Second-Degree Algebraic Inequalities – Solutions

CASE III : $\Delta < 0, a > 0$ → No real solution

- $ax^2 + bx + c > 0 \Rightarrow \forall x \in R$



- $ax^2 + bx + c < 0 \Rightarrow \nexists x \in R$



CASE III : $\Delta < 0, a > 0$ → No real solution

- $ax^2 + bx + c \geq 0 \Rightarrow \forall x \in R$



- $ax^2 + bx + c \leq 0 \Rightarrow \nexists x \in R$



Example. Solve the following inequality: $-x^2 + 12x - 11 \geq 0$

Solution: Change the sign of all terms and reverse the direction of the inequality:

$$x^2 - 12x + 11 \leq 0$$

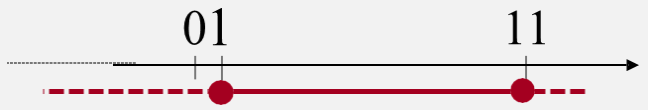
Compute the discriminant $\Delta = b^2 - 4ac = 144 - 44 = 100$

The associated equation admits two distinct real solutions:

$$x_{1,2} = \frac{12 \pm \sqrt{100}}{2} = \frac{12 \pm 10}{2} = \begin{matrix} 1 \\ 11 \end{matrix}$$

Based on the direction of the inequality and the sign of the discriminant, the solution set is:

$$1 \leq x \leq 11$$



Absolute Value Inequalities

Inequalities with a single absolute value with a positive number n at the right-hand side

- $|A| > n \rightarrow A < -n \vee A > n$
- $|A| \geq n \rightarrow A \leq -n \vee A \geq n$
- $|A| < n \rightarrow \begin{cases} A < n \\ A > -n \end{cases}$
- $|A| \leq n \rightarrow \begin{cases} A \leq n \\ A \geq -n \end{cases}$

Inequalities with a single absolute value with a negative number $-n$ at the right-hand side

- $|A| > -n \rightarrow \forall x \in \mathbb{R}$
- $|A| \geq -n \rightarrow \forall x \in \mathbb{R}$
- $|A| < -n \rightarrow$ nessuna soluzione
- $|A| \leq -n \rightarrow$ nessuna soluzione

Inequalities with a single absolute value with zero at the right-hand side

- $|A| > 0 \rightarrow A \neq 0$
- $|A| \geq 0 \rightarrow \forall x \in \mathbb{R}$
- $|A| < 0 \rightarrow$ nessuna soluzione
- $|A| \leq 0 \rightarrow A = 0$

REMEMBER

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $(x+y)(x-y) = x^2 - y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Remember: Logarithms

the logarithm of a number is the exponent x to be given to the base a to obtain the argument b → $a^x = b$

$$log_a(b) = x$$

a = base → must be $a > 0$ and $a \neq 1$
b = argument → must be $b > 0$
x = logarithm of b in base a → is $\in \mathbb{R}$

Properties: $log_a(a) = 1$ | $log_a(1) = 0$ | $a^x > 0$

Some theorems:

$log_a(b \cdot c) = log_a(b) + log_a(c)$	$log_2(3 \cdot x) = log_2(3) + log_2(x)$
$log_a\left(\frac{b}{c}\right) = log_a(b) - log_a(c)$	$log_2\left(\frac{x}{3}\right) = log_2(x) - log_2(3)$
$log_a(b)^c = c\, log_a(b)$	$log_2(x)^3 = 3\, log_2(x)$

Example: pH is a logarithm

Hydrogen-ion concentration spans many orders of magnitude, so we compress it with a logarithm:

$$pH = -\log_{10}[H^+]$$

A fall of one pH unit means ten times more H^+.

Blood is held tightly near pH 7.4; small shifts are clinically serious.

Logarithms turn a huge multiplicative range into a simple additive scale.

