APPLIED PHYSICS AND BIOSTATISTICS

MODULE MATH3 — Elementary Functions and Biomedical Models

Elementary functions & biomedical models

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Each Elementary Function Models Something in Biology

The few elementary functions are the building blocks of every biomedical model.

Linear → calibration, Beer–Lambert.

Power → allometric scaling (metabolism ∝ mass^0.75).

Exponential → growth of cells and viruses; decay of drugs and tracers (half-life).

Logarithm → pH, decibels, log dose–response — compressing huge ranges.

Trigonometric (periodic) → heartbeat, breathing, circadian rhythms.

Elementary Functions

Linear Function

Linear Function

A linear function is defined as a function whose general form is:

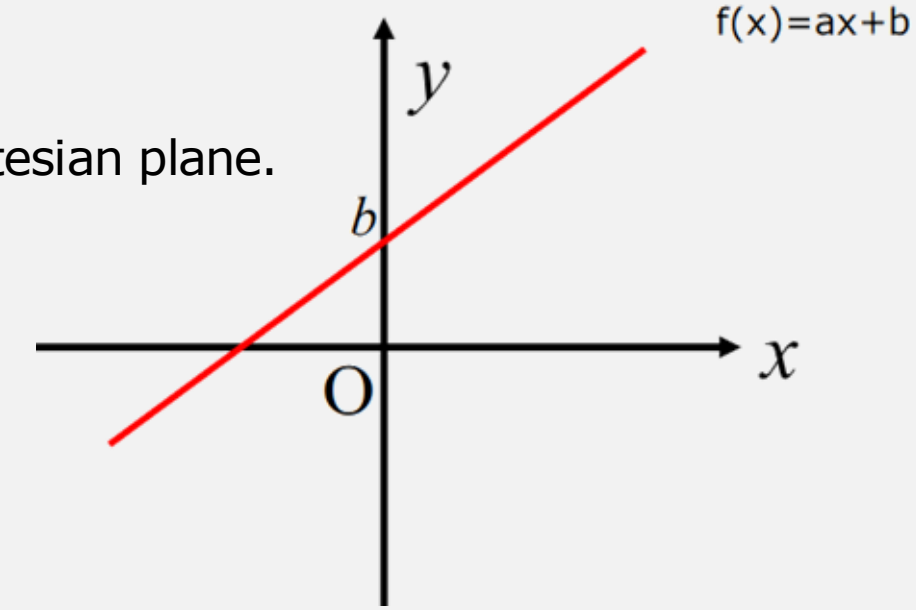
$$f(x) = ax + b, \qquad a, b \in \mathbb{R}$$

$$f : x \in \mathbb{R} \rightarrow (ax + b) \in \mathbb{R}$$

The constant a is called the slope (or angular coefficient) of the function and represents the rate of change of the function with respect to the independent variable.

The constant b is called the y-intercept (or constant term).

The graph of any linear function is a straight line in the Cartesian plane.



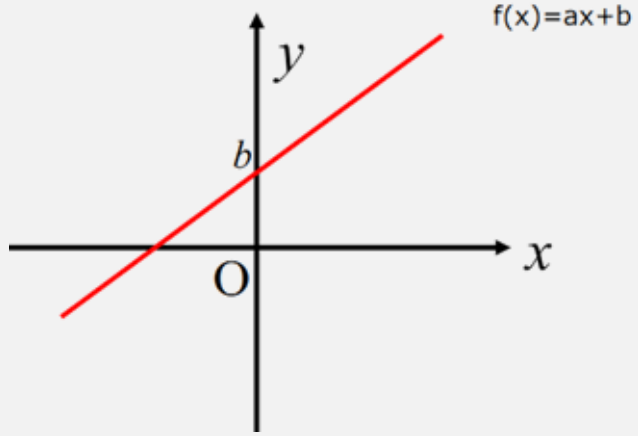
The Constant Term b

A linear function is characterized by two constants, a and b.

If $x = 0 \rightarrow y = b$, therefore the point $P(0, b)$ which lies on the vertical y-axis, belongs to the graph of the function.

The constant term b is the ordinate of the intersection of the graph of the function with the y-axis and is called the INTERCEPT.

If the independent variable is time and the function is written as $f(t) = at + b$, the coefficient $b = f(0)$ is called the INITIAL VALUE

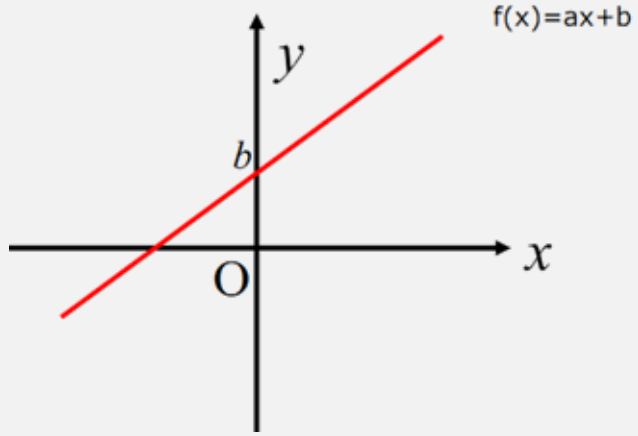


The Slope a

Given the function $f(x) = ax + b$, the coefficient a represents the variation of the ordinates of two arbitrary points on the graph with respect to the corresponding variation of the abscissas:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- The coefficient a quantifies the inclination of the straight line with respect to the horizontal axis.
- The coefficient a is constant.

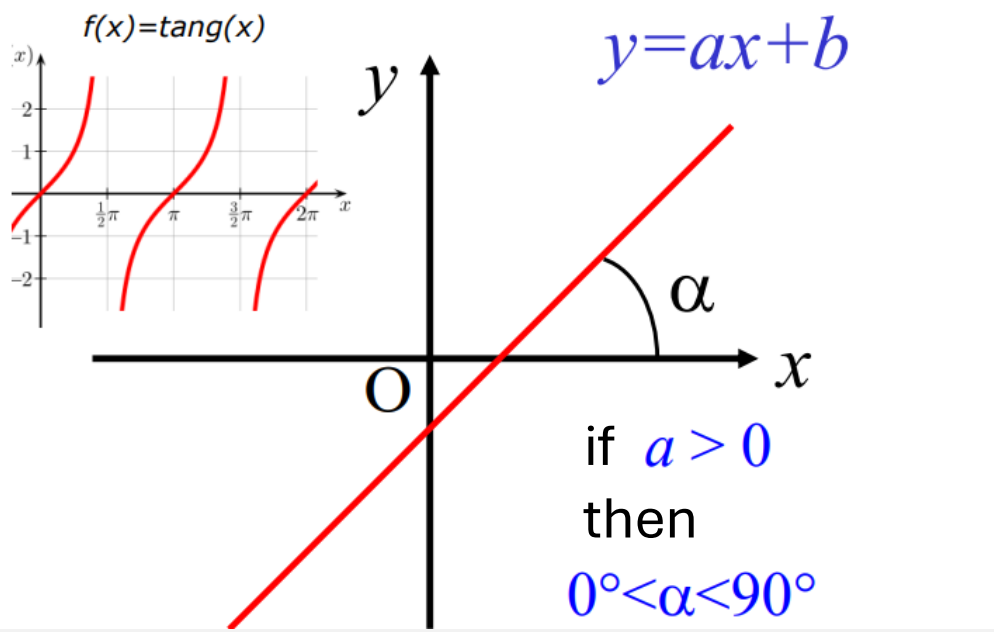


The Slope a

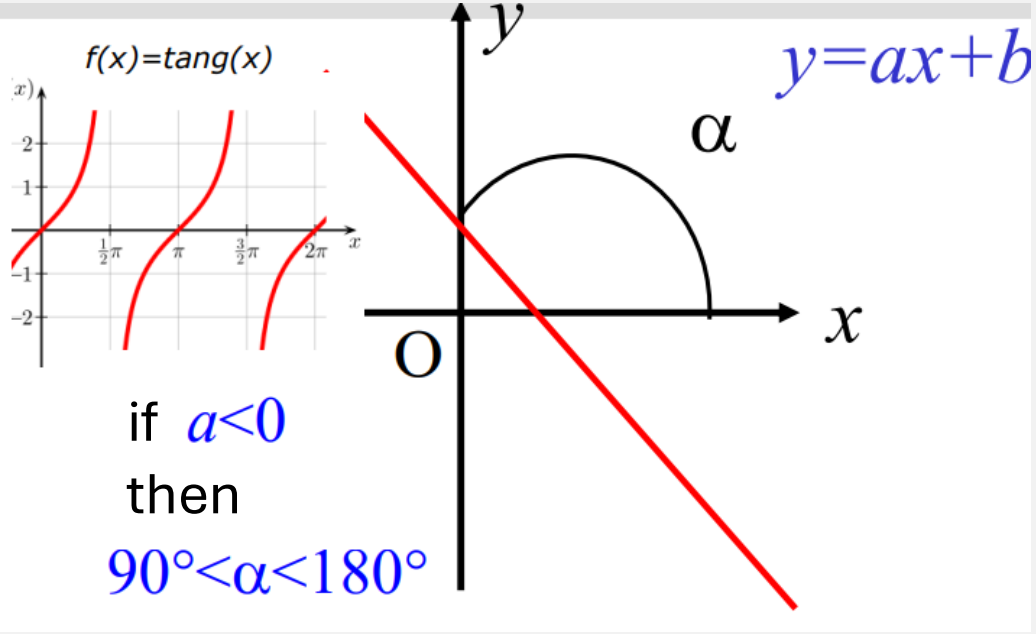
The slope a is given by:

$$a = \tan \alpha$$

where α is the angle formed by the line and the horizontal axis, measured counterclockwise



f strictly increasing in $\mathbb{R}$
$\sup(ax + b) = +\infty;\ \inf(ax + b) = -\infty$



f strictly decreasing in $\mathbb{R}$
$\sup(ax + b) = +\infty;\ \inf(ax + b) = -\infty$

Elementary Functions

Absolute Value Function

The Absolute Value Function

The absolute value function is defined by:

$$f(x) = |x|$$

$$f : x \in \mathbb{R} \to |x| \in [0, +\infty)$$

where

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

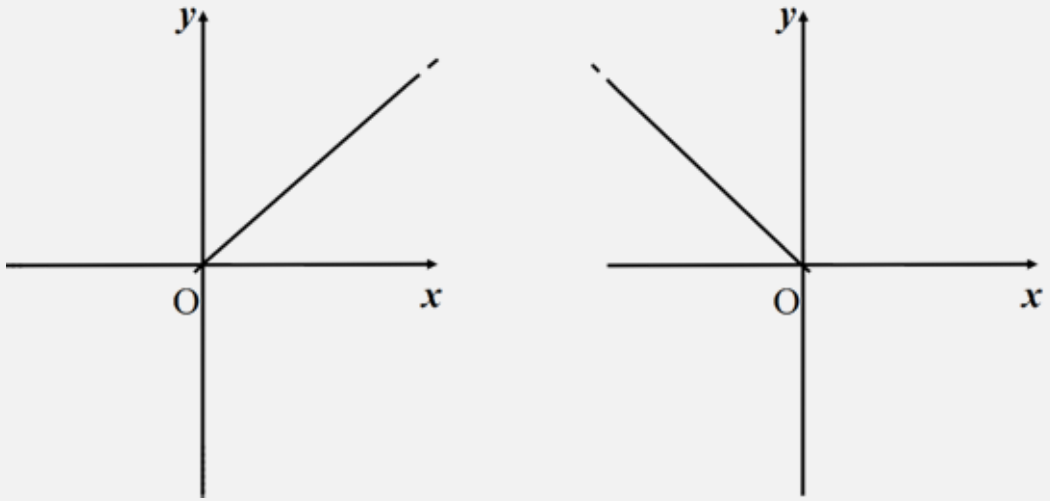
Its graph consists of two half-lines passing through the origin.

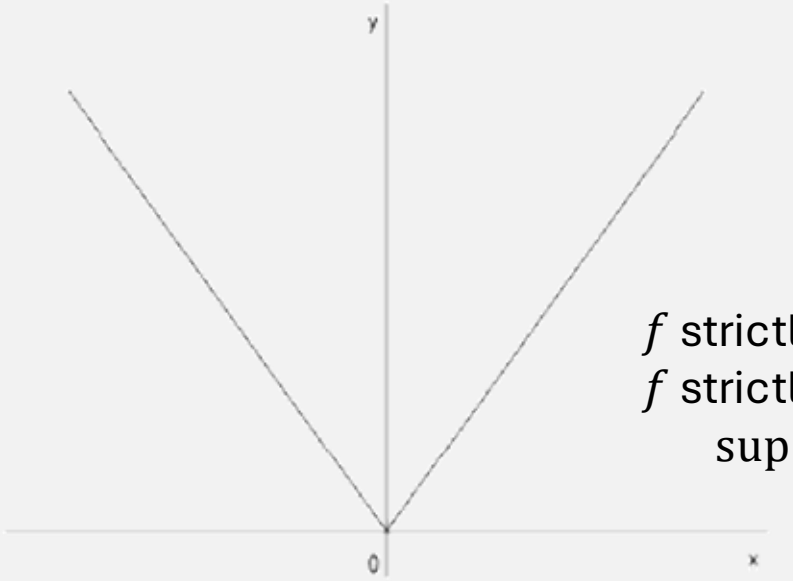
$$f(x) = x \text{ and } f(x) = -x$$

$$f(x) = |x|$$

$$f : x \in \mathbb{R} \rightarrow |x| \in [0, +\infty)$$

$$f(x) = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$





f strictly decreasing in $(-\infty, 0]$
f strictly increasing in $[0, +\infty)$
$\sup|x| = +\infty;\ \inf|x| = 0$

Elementary Functions

Power Function

The Power Function with Natural Exponent

$$f(x) = x^n, \qquad n \in \mathbb{N} \text{ fixed}$$

$$f : x \in \mathbb{R} \to x^n \in codomain\ depending\ on\ n$$

x is the base of the power function and varies in the domain
n is the exponent of the power function and is fixed

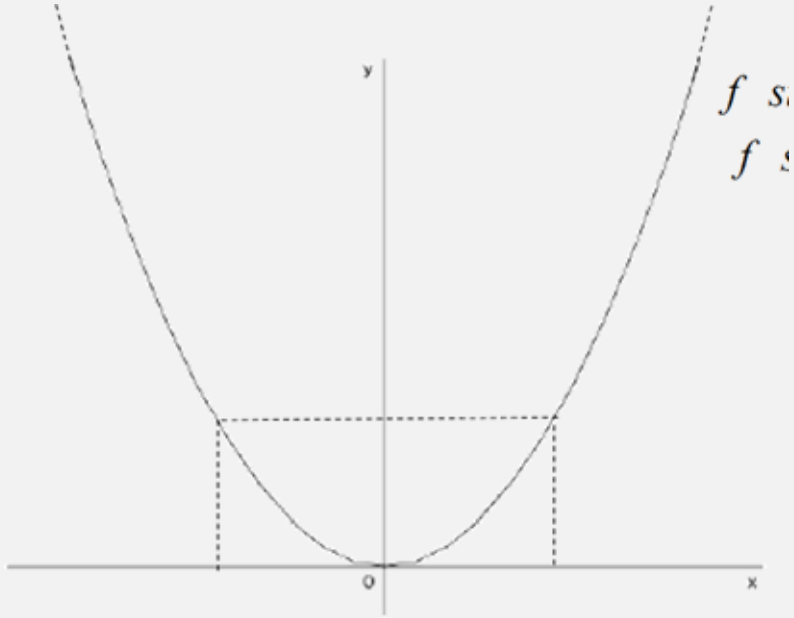
Rules:

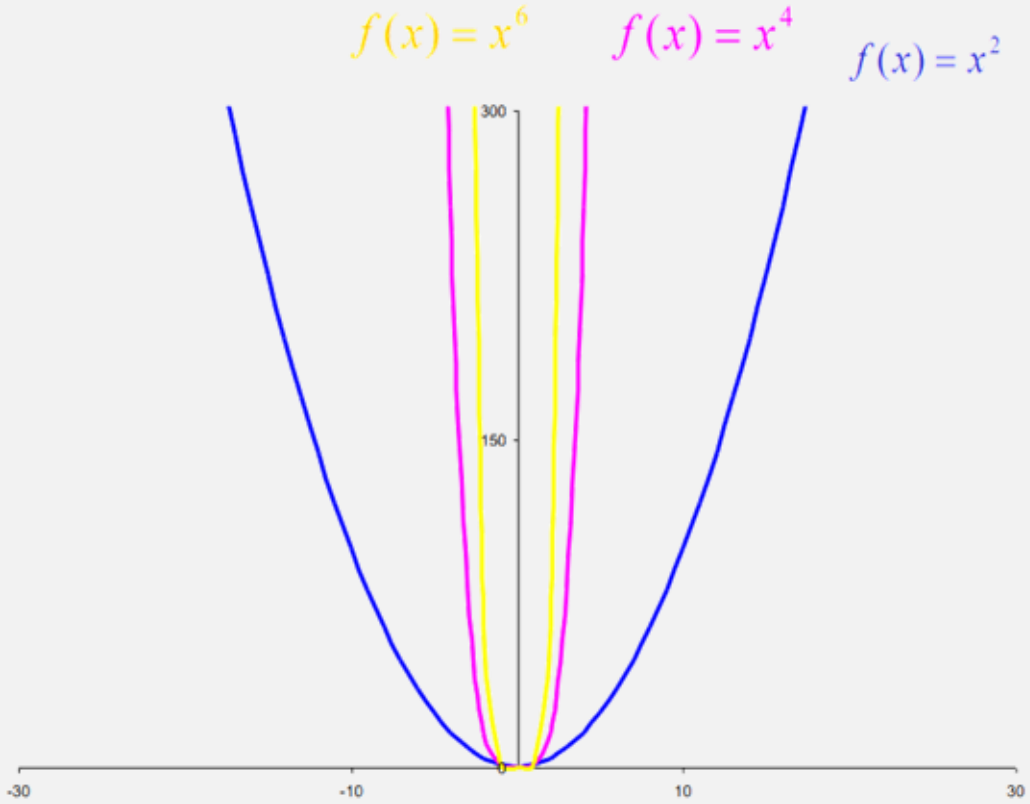
Positive base: any exponent is admissible
Negative base: integer exponent or rational exponent with odd denominator

The Power Function with Even Natural Exponent

$$f(x) = x^n, \qquad n \in \mathbb{N} \text{ fixed and even}$$

$$f : x \in \mathbb{R} \to x^n \in [0, +\infty)$$



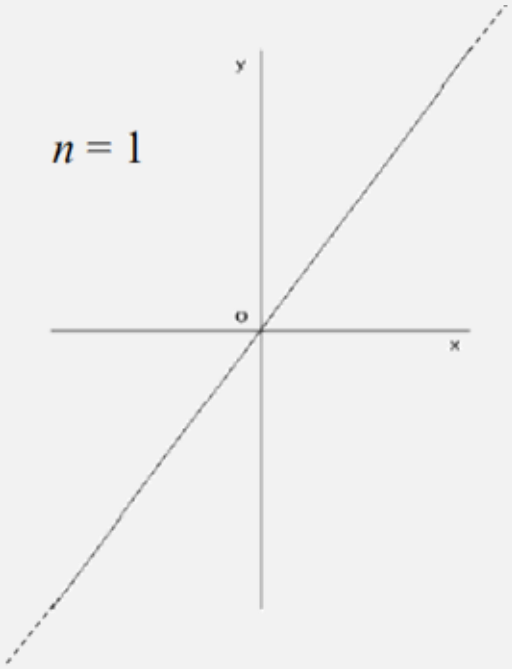


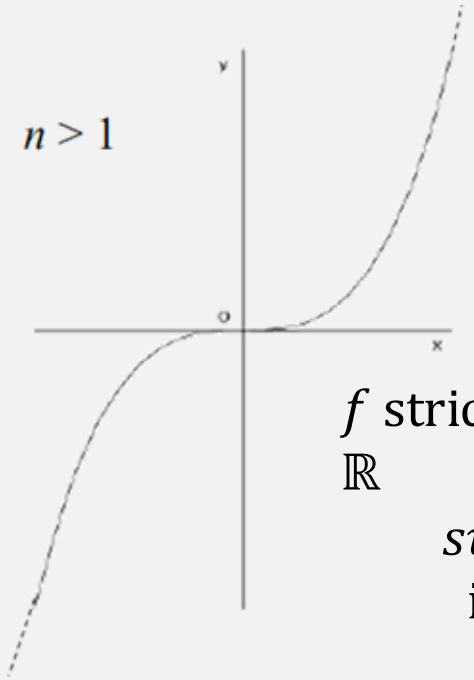
f strictly decreasing in $(-\infty, 0)$
f strictly increasing in $(0, +\infty)$
$sup\, x^n = +\infty;\ \inf x^n = 0$

The Power Function with Odd Natural Exponent

$$f(x) = x^n, \qquad n \in \mathbb{N} \text{ fixed and odd}$$

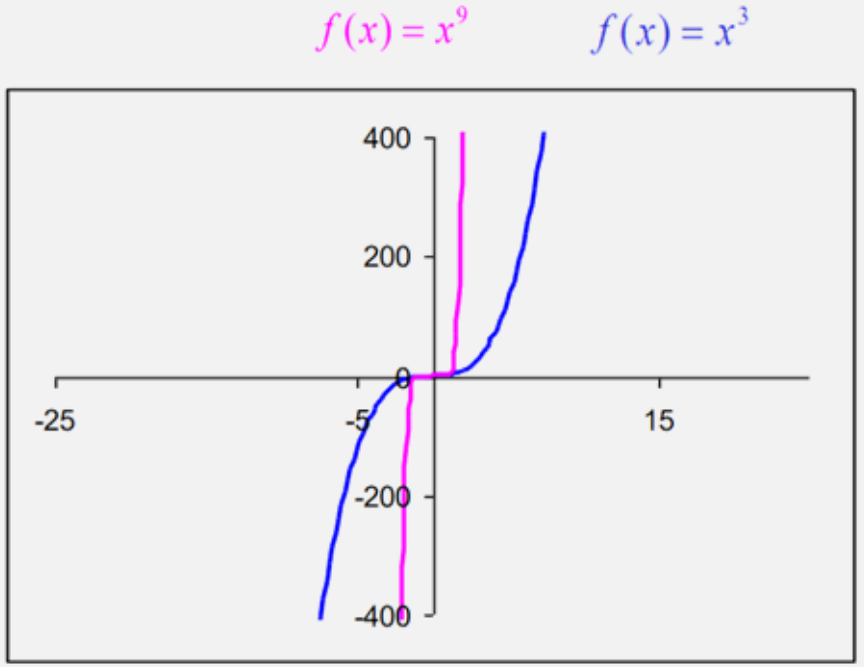
$$f : x \in \mathbb{R} \to x^n \in \mathbb{R}$$





f strictly increasing in $\mathbb{R}$

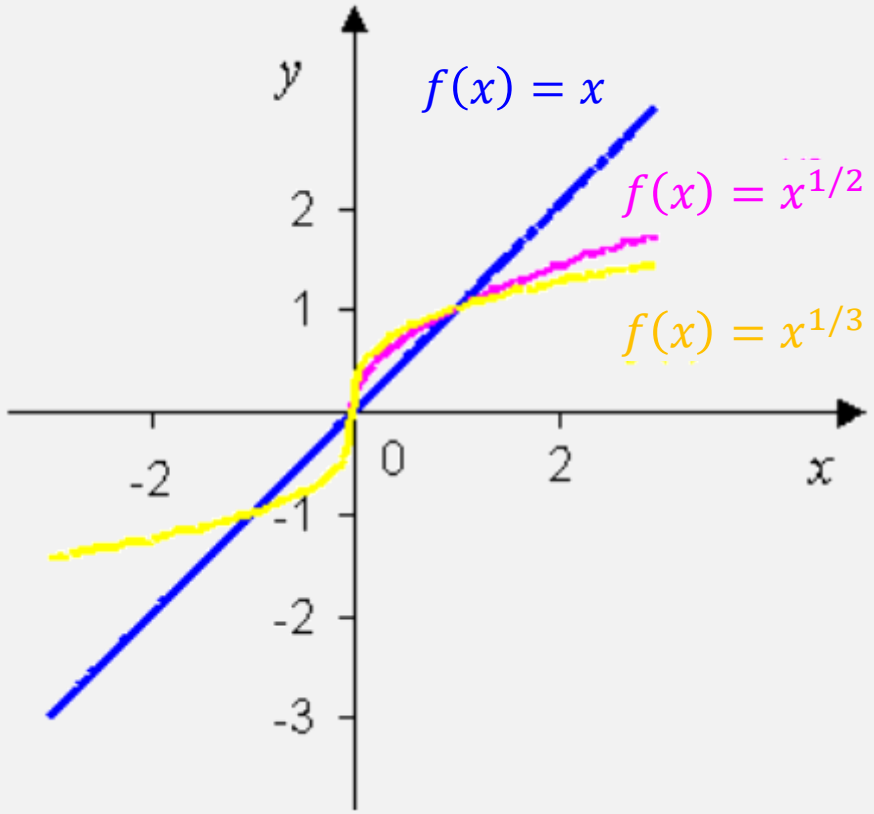
$$sup\ x^n = +\infty;$$
$$\inf x^n = -\infty$$



The Power Function with Exponent $1/n$ (Root Function)

$$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}, \qquad n \in \mathbb{N} \text{ fixed}, n \geq 2$$

The domain and codomain depend on the value of n



Elementary Functions

Polynomial Function

The Polynomial (i.e. Quadratic) Function

A polynomial function is defined as a (generally non-monotonic) function expressed by the general form:

$$f(x) = a_n x^n + \cdots + a_1 x + a_0$$
$$f : D \to \mathbb{R}$$

$n \in \mathbb{N}$ is the degree of the polynomial

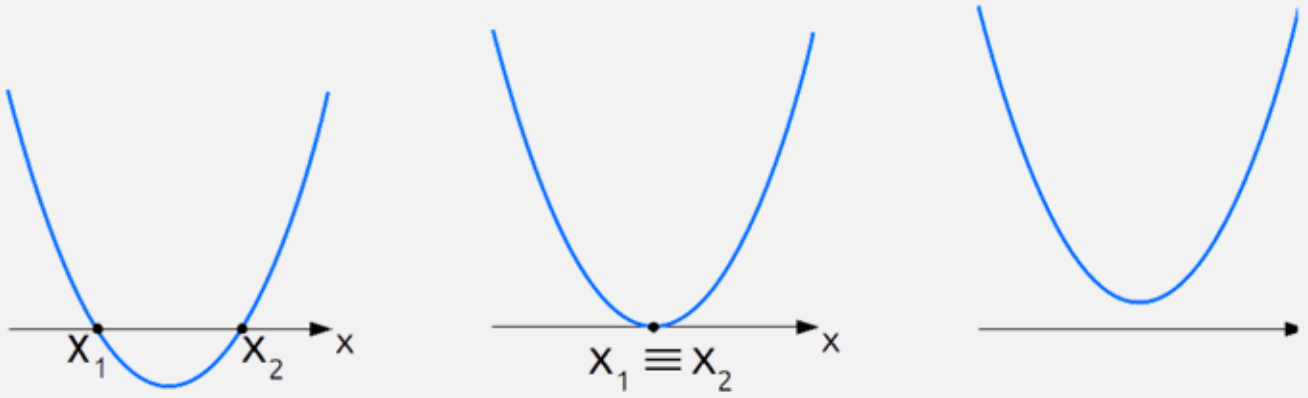
$a_n, \ldots, a_1, a_0 \in \mathbb{R}$ are the coefficients

The leading coefficient a_n is $\neq 0$

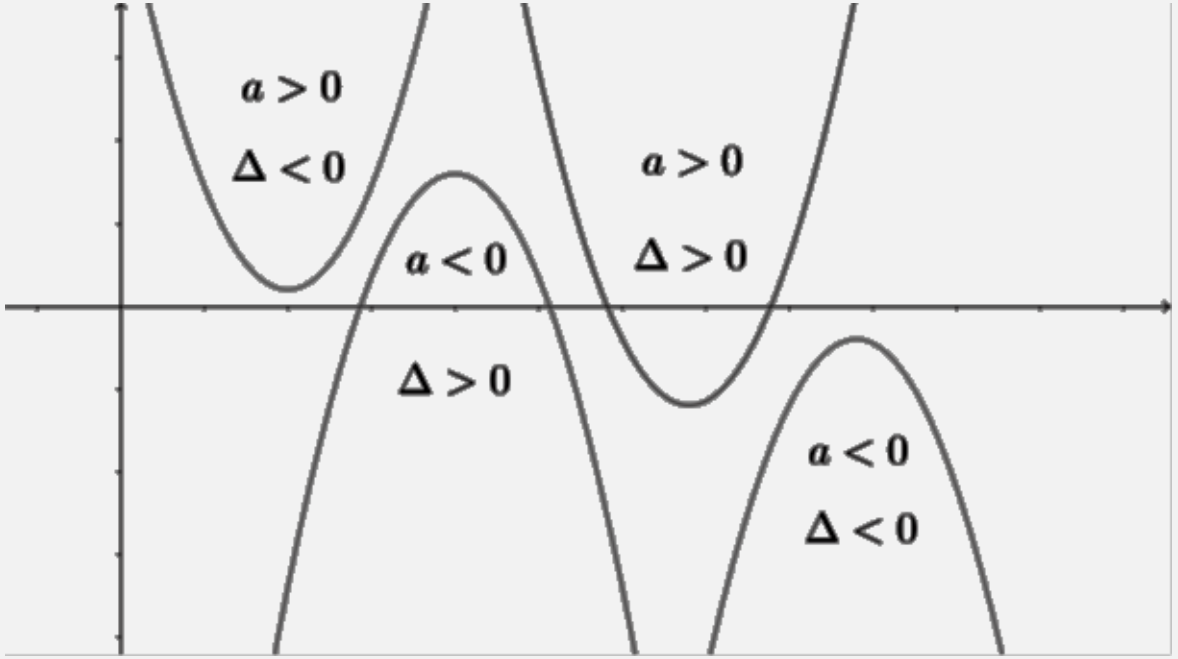
If the degree of the polynomial defining f is equal to 2:

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

- if $a > 0$, the parabola opens upwards, wihile if $a < 0$, the parabola opens downwards
- the abscissa of the vertex is $x = -\frac{b}{2a}$
- the vertex is $V = (-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$ and represents the symmetry axis of the parabola
- if $a > 0$ the ordinate of the vertex represents the minimum value of the function $f(x) = ax^2 + bx + c$, while if $a < 0$ it represents the maximum value
- the point $(0, c)$ is the intersection with the y-axis.
- The solutions of the equation $ax^2 + bx + c = 0$ give the intersections with the x-axis.



$\Delta > 0$ $\Delta = 0$ $\Delta < 0$



Elementary Functions

Exponential Function

The Exponential Function

$$f(x) = a^x, a \in \mathbb{R}\,\text{fixed}, a > 0, a \neq 1$$

$$f : x \in \mathbb{R} \to a^x \in (0, +\infty)$$

a is the base of the exponential function and is fixed
x is the exponent and varies in the domain

The exponential function is always positive

The Exponential Function with Base $a > 1$

$$f(x) = a^x, a \in \mathbb{R}, a > 0, a \neq 1$$

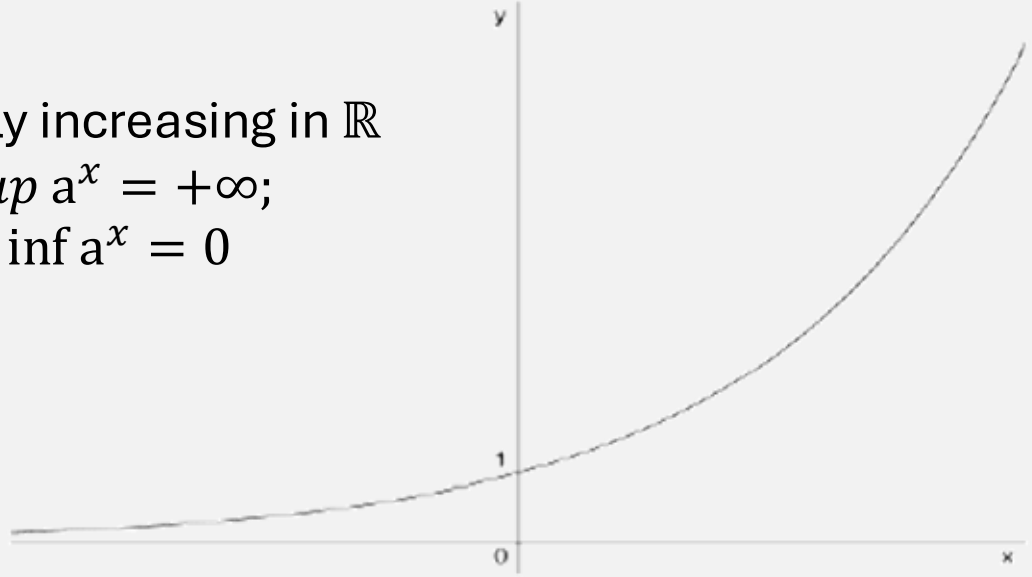
$$\text{case } a > 1$$

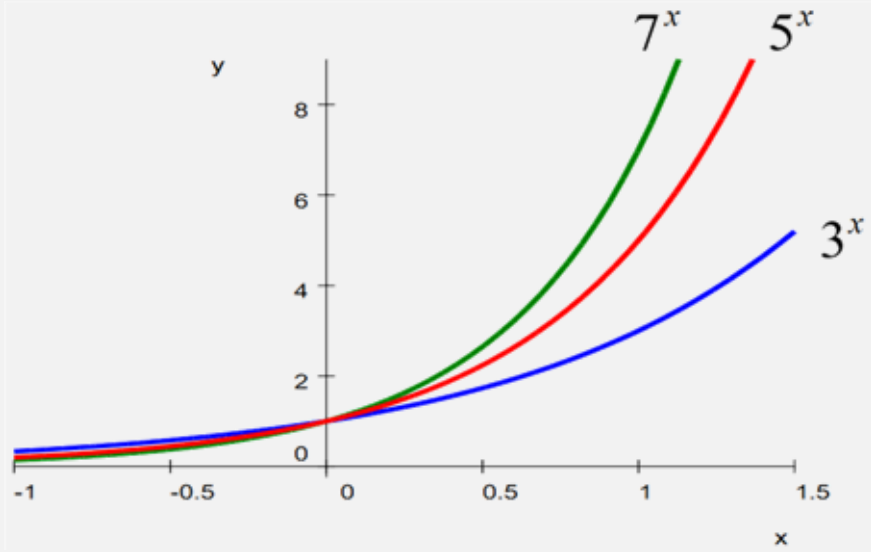
$$f : x \in \mathbb{R} \to a^x \in (0, +\infty)$$

f strictly increasing in $\mathbb{R}$

$sup\ a^x = +\infty;$

$\inf a^x = 0$



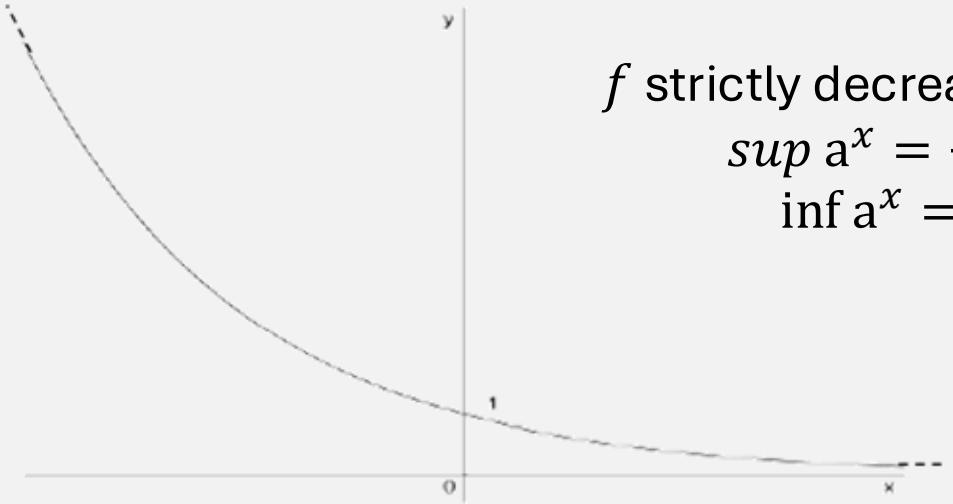


The Exponential Function with Base $0 < a < 1$

$$f(x) = a^x, a \in \mathbb{R}, a > 0, a \neq 1$$

$$\text{case } 0 < a < 1$$

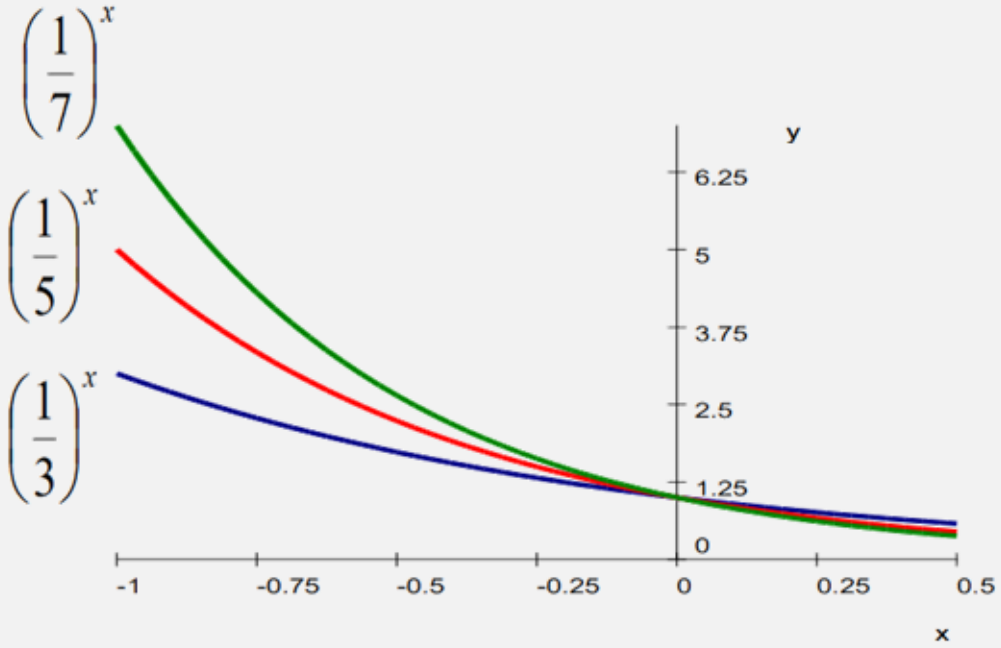
$$f : x \in \mathbb{R} \to a^x \in (0, +\infty)$$



f strictly decreasing in $\mathbb{R}$

$sup\ a^x = +\infty;$

$\inf a^x = 0$



The Exponential Function with Base e

A particularly important exponential function has as base the irrational number

$$e = 2.7182$$

called Napier's number

The exponential function $f(x) = e^x$ is strictly increasing in $\mathbb{R}$ since $e > 1$

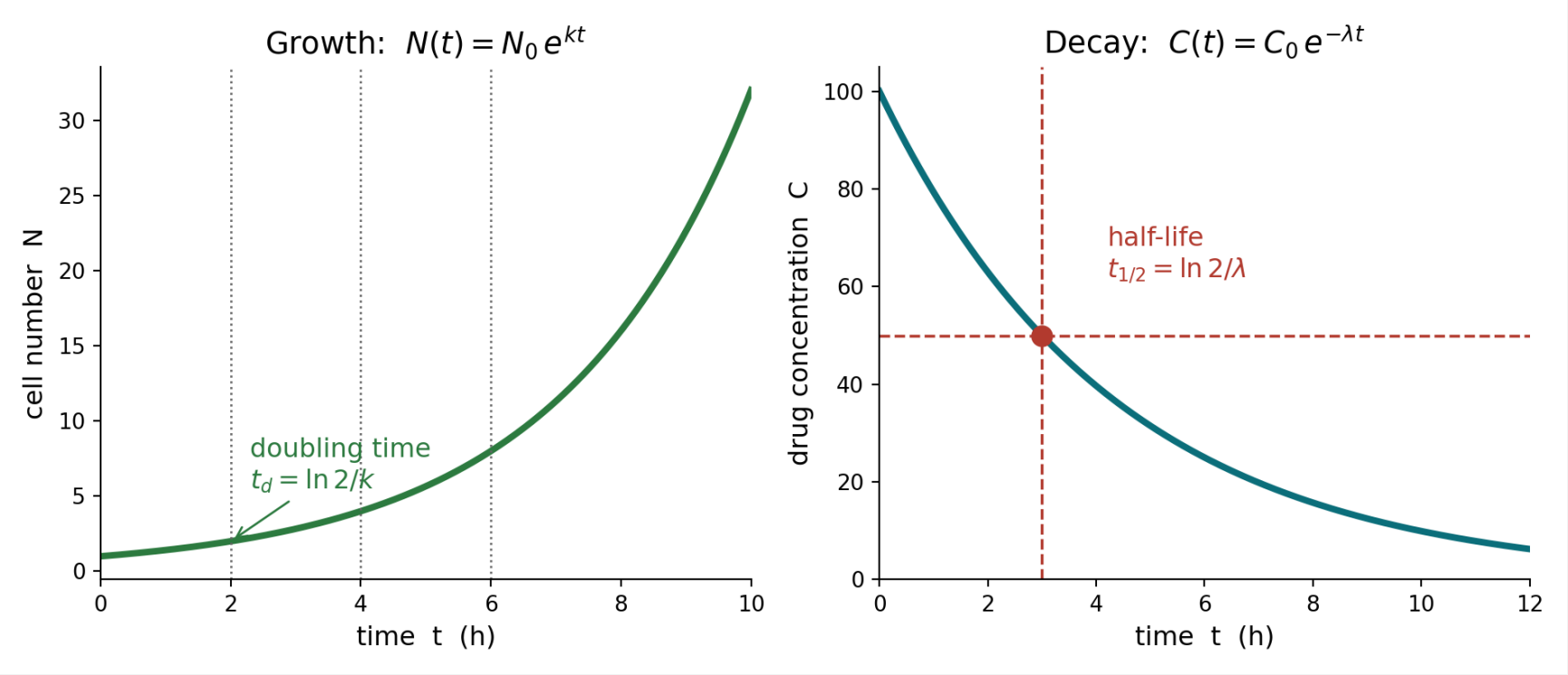
Example - Exponential Growth and Decay

The same shape describes opposite biology, according to the sign of the exponent:

Growth (bacteria, viruses): $N(t) = N_0 e^{(kt)}$, doubling time $t_d = ln2/k$

Decay (drug elimination, tracer): $C(t) = C_0 e^{(-\lambda t)}$, half-life $t(1/2) = ln2/\lambda$

A constant doubling time (or half-life) is the signature of an exponential process.



Elementary Functions

Logarithmic Function

The Logarithmic Function

$$f(x) = \log_a x\,, a \in \mathbb{R}\,\text{fixed}, a > 0, a \neq 1$$

$$f : x \in (0, +\infty) \to \log_a x \in \mathbb{R}$$

a is the base of the logarithm
x is the argument and varies within the domain

$$a^x = b \Leftrightarrow x = \log_a b$$

The logarithm is the inverse function of the exponential:

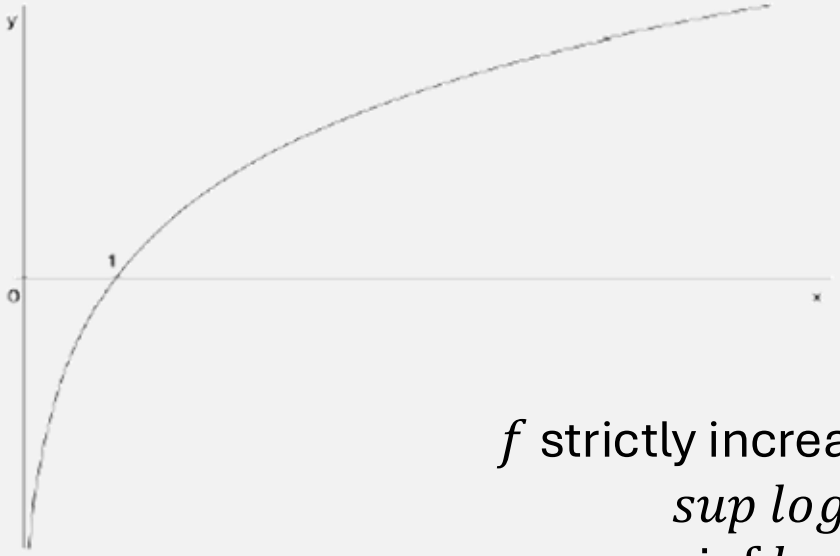
$$x = a^{\log_a x} \qquad x = \log_a(a^x)$$

The Logarithmic Function with Base $a > 1$

$$f(x) = \log_a x\,, a \in \mathbb{R}, a > 0, a \neq 1$$

$$\text{case } a > 1$$

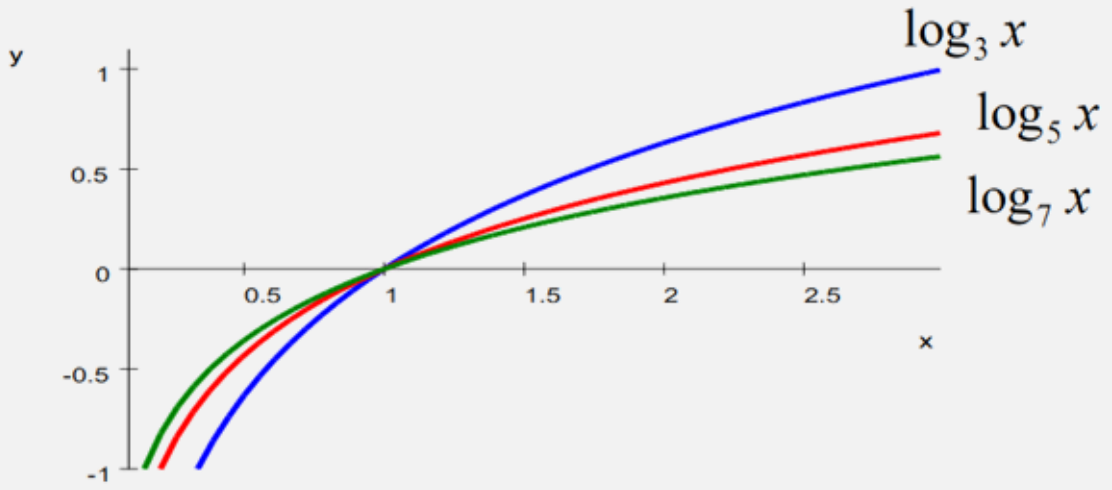
$$f : x \in (0, +\infty) \to \log_a x \in \mathbb{R}$$



f strictly increasing in $(0, +\infty)$

$sup\ log_a \mathrm{x} = +\infty;$

$\inf log_a \mathrm{x} = -\infty$

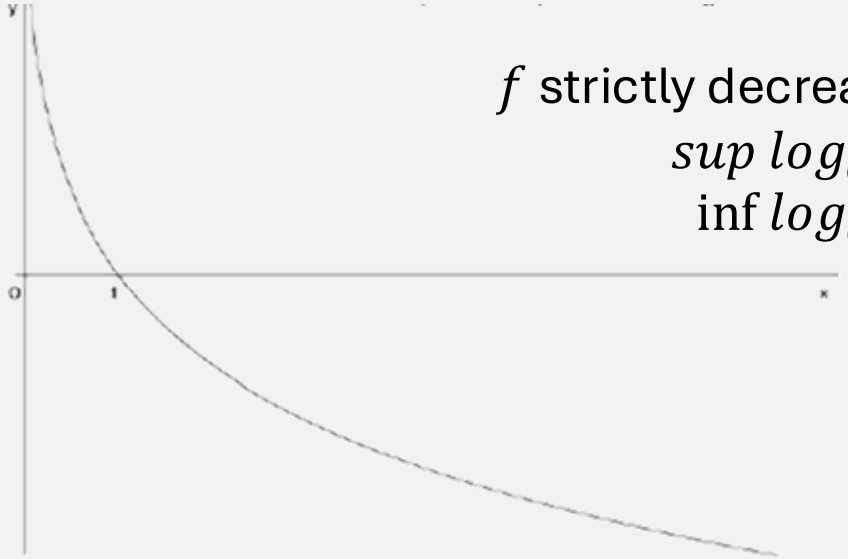


The Logarithmic Function with Base $0 < a < 1$

$$f(x) = \log_a x\,, a \in \mathbb{R}, a > 0, a \neq 1$$

$$\text{case } 0 < a < 1$$

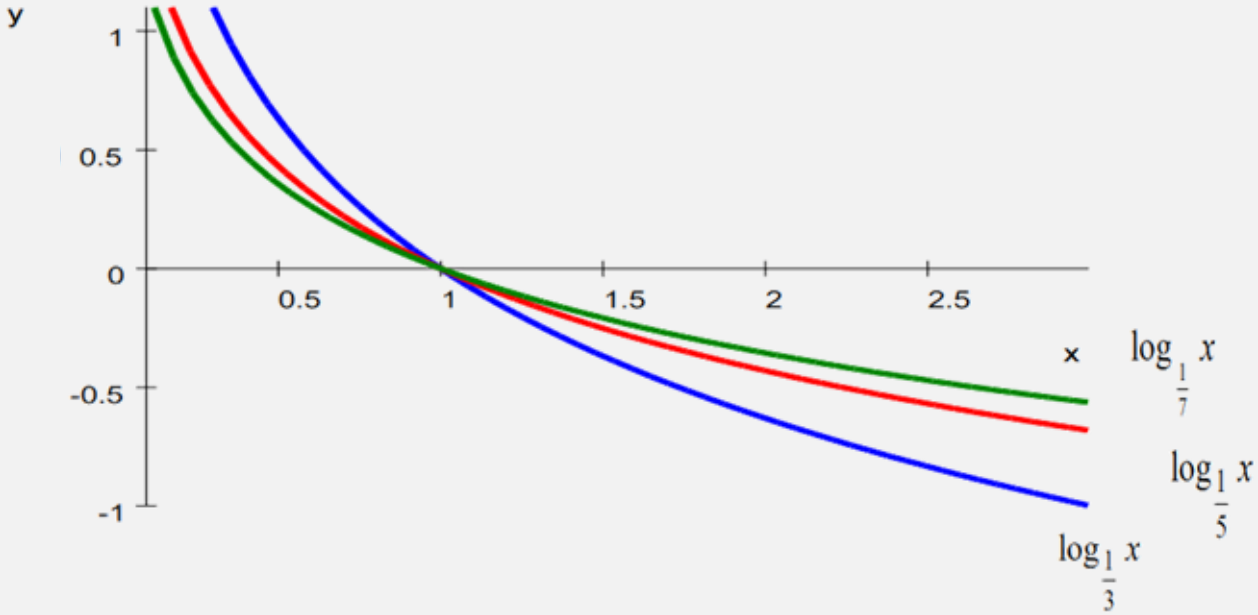
$$f : x \in (0, +\infty) \to \log_a x \in \mathbb{R}$$



f strictly decreasing in $(0, +\infty)$

$sup\ log_a \mathrm{x} = +\infty;$

$\inf log_a \mathrm{x} = -\infty$



The Exponential Function with Base e (or Natural Logarithm)

The logarithmic function with base e is called the natural logarithm:

$$f(x) = \log_e x = \log x = \ln x$$

The exponential function $f(x) = \log_e x$ is a strictly increasing function because $e > 1$

Example – Dose-Response on a Logarithmic Scale

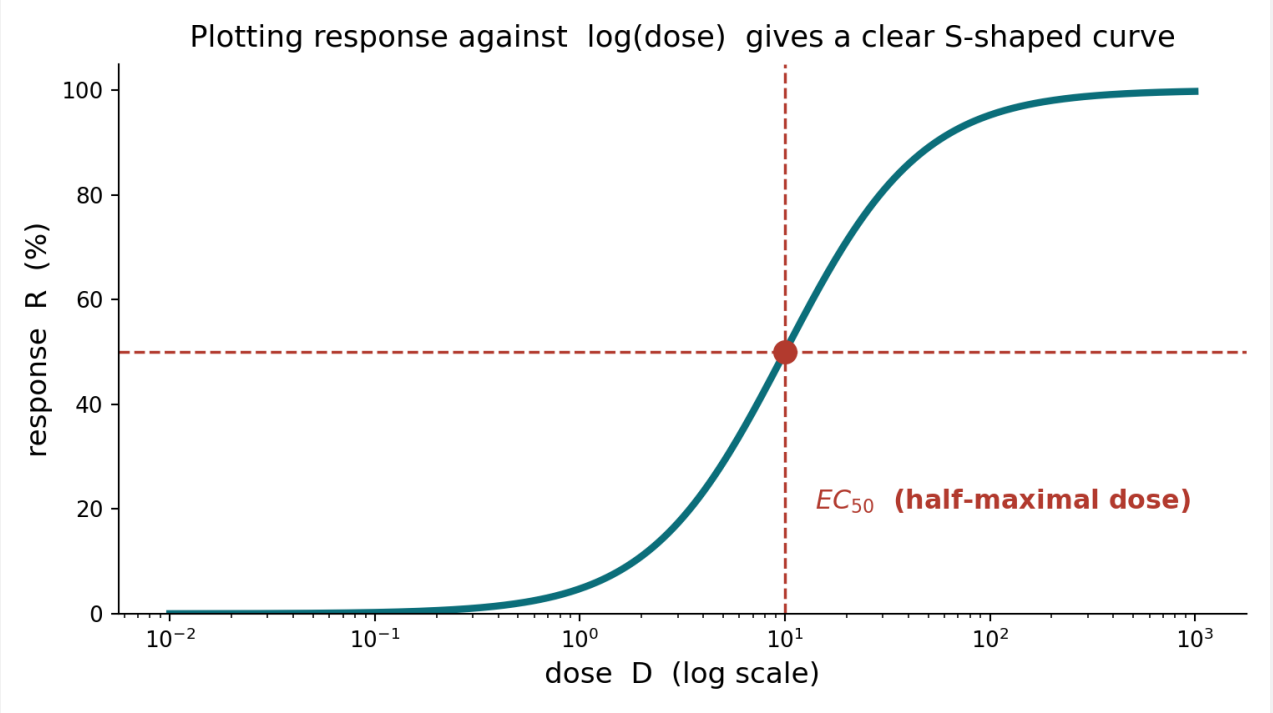
Drug doses span orders of magnitude, so we plot response against log(dose).

On this scale the response becomes a clear S-shaped (sigmoid) curve.

The dose giving half the maximal response is the potency marker EC_{50}.

$$R = E_{max} \cdot D^{h} / (EC_{50}{}^{h} + D^{h})$$

Logarithms make a wide dose range comparable and reveal EC_{50} at a glance.

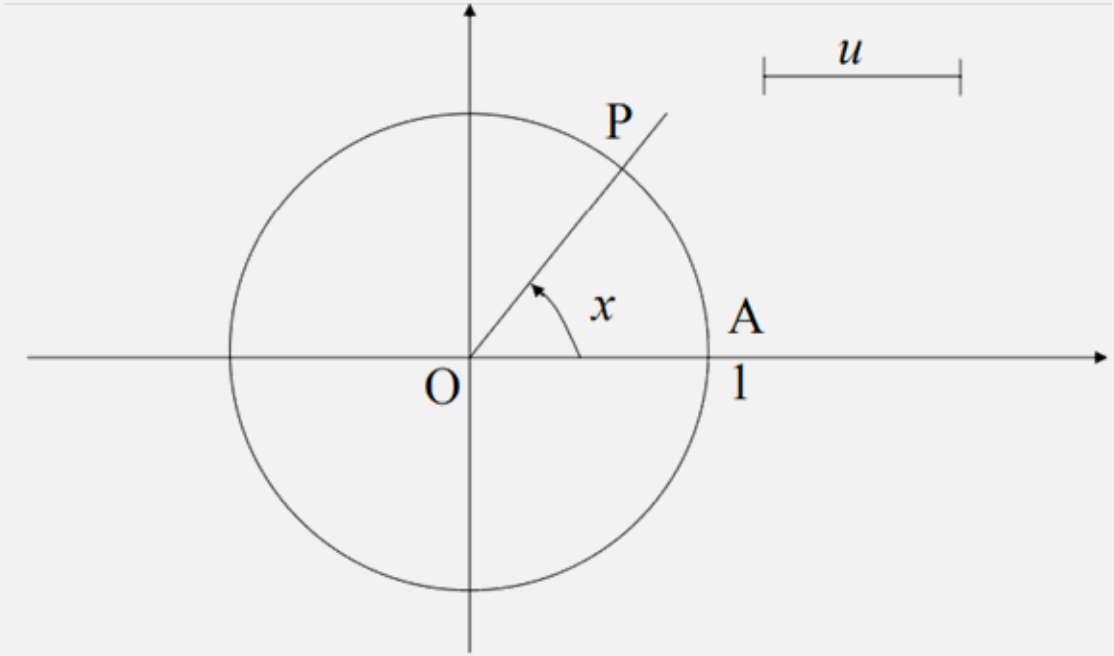


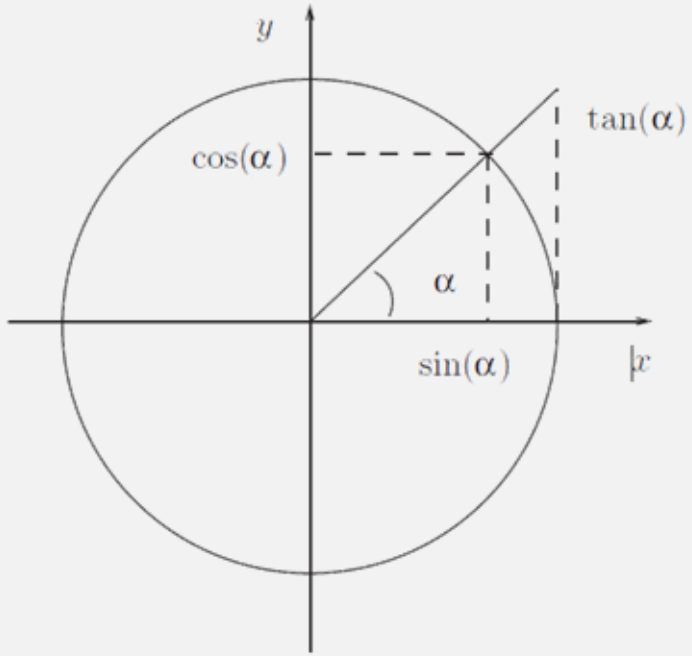
Elementary Functions

Trigonometric Functions

The Unit Circle

Let P be a point moving on the unit circle, starting from point A, origin of arcs, and let x be the angle subtended by P.

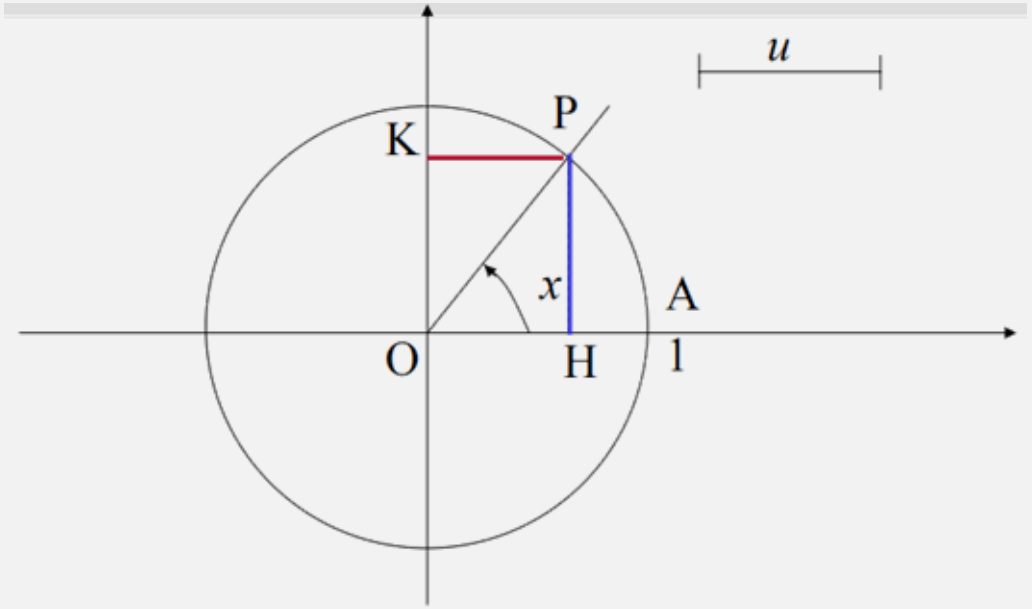


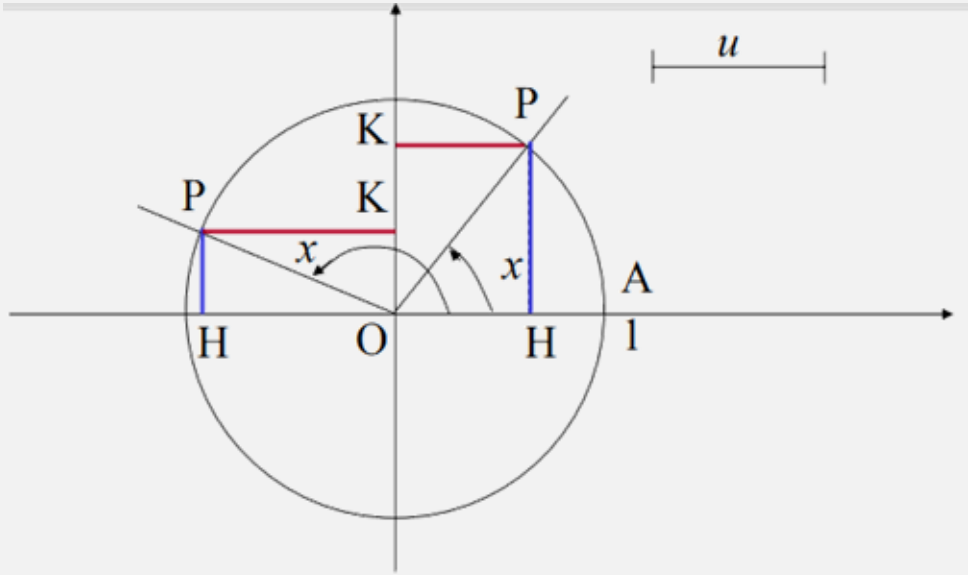


Fundamental identity of trigonometry: $\cos^2 x + \sin^2 x = 1$

Sine and Cosine

$\sin x$ is defined as the ordinate of point P (meaning the segment PH).
$\cos x$ is defined as the abscissa of point P (meaning the segment PK).





As the position of P varies, the angle x, and hence $\sin x$ and $\cos x$, vary.

Sine Function

$$f(x) = \sin x$$

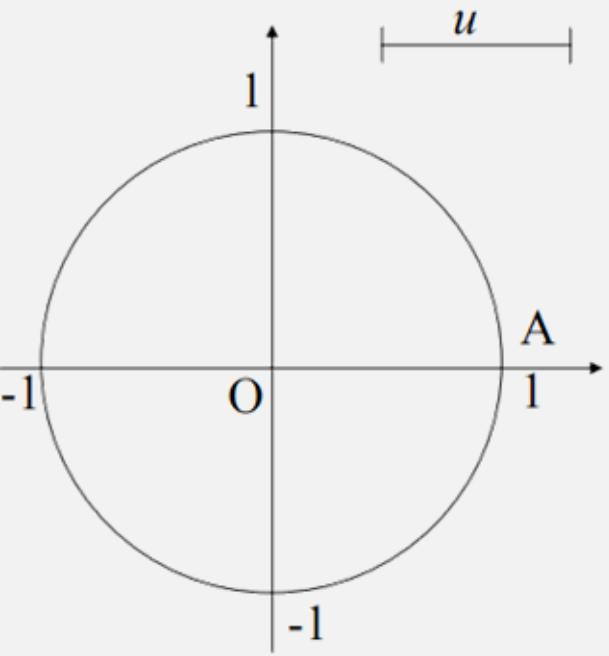
$$f : x \in \mathbb{R} \rightarrow \sin x \in [-1, +1]$$

The domain of the sine function is the whole set $\mathbb{R}$, since a point P moving on the unit circle can traverse it infinitely many times, performing infinitely many revolutions (at each revolution, the angle increases by an amount equal to 2π).

The codomain of the sine function is the interval $[-1, +1]$, since the minimum value attained by the ordinate of point P is -1, while the maximum value is $+1$.

Sine Function

For each fixed value of the arc x, the sine function assumes a corresponding numerical value.



x	*sen x*
0	0
π/4	$\frac{\sqrt{2}}{2}$
π/2	1
π	0
π/6	1/2
π/3	$\frac{\sqrt{3}}{2}$
3/2 π	-1
2 π	0

Periodicity

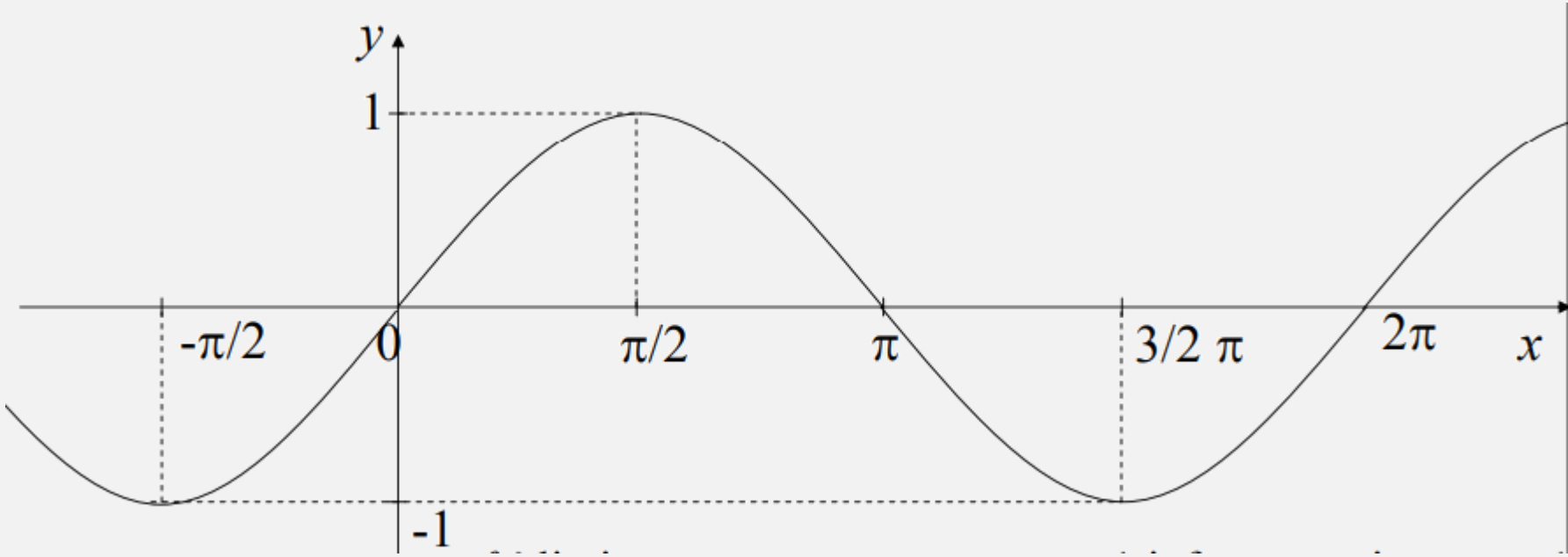
At each complete revolution (corresponding to an arc of legth 2π), the function $\sin x$ takes the same values

→ therefore, the sine function is <u>periodic</u> with period 2π:

$$\sin(x + 2\pi) = \sin x\,, \forall x \in \mathbb{R}$$

Notable points

- $(0,0) \rightarrow f(0) = 0$
- $\left(\frac{\pi}{2}, 1\right) \rightarrow f\left(\frac{\pi}{2}\right) = 1$
- $(\pi, 0) \rightarrow f(\pi) = 0$
- $\left(\frac{3\pi}{2}, -1\right) \rightarrow f\left(\frac{3\pi}{2}\right) = -1$
- $(2\pi, 0) \rightarrow f(2\pi) = 0$



f is bounded
f is strictly increasing in $[-\pi/2 + 2k\pi, \pi/2 + 2k\pi], k \in \mathbb{Z}$
f is strictly decreasing in $[\pi/2 + 2k\pi, 3/2\pi + 2k\pi], k \in \mathbb{Z}$

Cosine Function

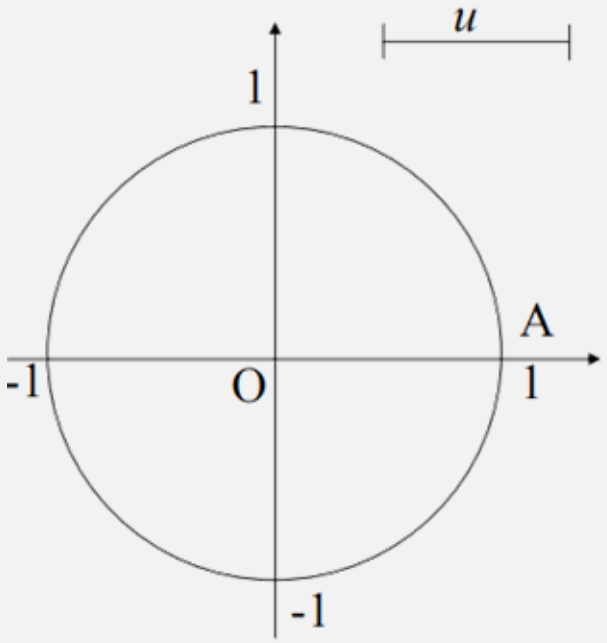
$$f(x) = \cos x$$

$$f : x \in \mathbb{R} \rightarrow \cos x \in [-1, +1]$$

The domain of the cosine function is the whole set $\mathbb{R}$, since a point P moving on the unit circle can traverse it infinitely many times, performing infinitely many revolutions (at each revolution, the angle increases by an amount equal to 2π).

The codomain of the cosine function is the interval $[-1, +1]$, since the minimum value assumed by the ordinate of point P is -1, while the maximum value is $+1$.

For each fixed value of the arc x, the cosine function assumes a corresponding numerical value.

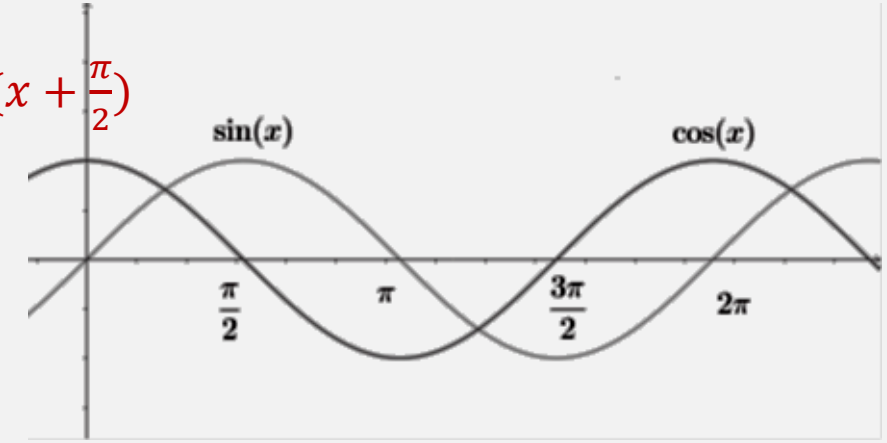


x	$\cos x$
0	1
π/4	$\frac{\sqrt{2}}{2}$
π/2	0
π	-1
π/6	$\frac{\sqrt{3}}{2}$
π/3	1/2
3/2 π	0
2 π	1

Periodicity
At each complete revolution (corresponding to an arc of length 2π), the function cos x takes the same values.
→ therefore, the cosine function is **periodic** of period 2π:

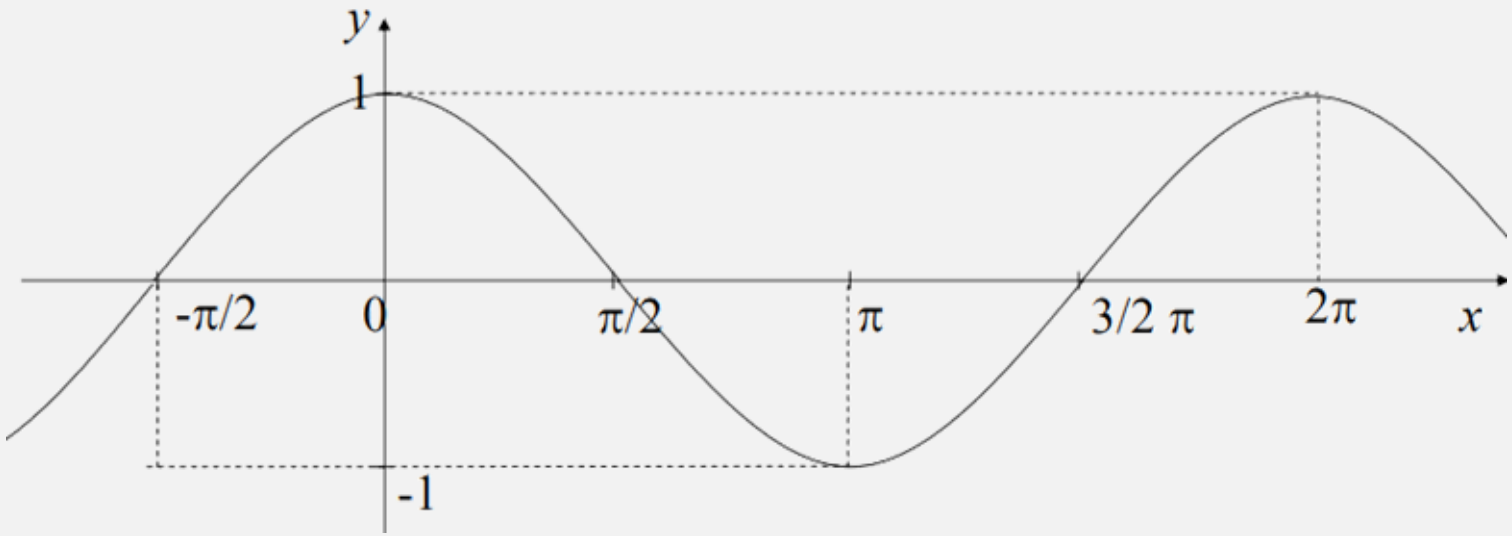
$$\cos(x + 2\pi) = \cos x\,, \forall x \in \mathbb{R}$$

The $\cos x$ function is equal to the $\sin x$ function shifted by $\frac{\pi}{2}$: $\cos x = \sin(x + \frac{\pi}{2})$



Notable points

- $(0,1) \rightarrow f(0) = 1$
- $\left(\frac{\pi}{2}, 0\right) \rightarrow f\left(\frac{\pi}{2}\right) = 0$
- $(\pi, -1) \rightarrow f(\pi) = -1$
- $\left(\frac{3\pi}{2}, 0\right) \rightarrow f\left(\frac{3\pi}{2}\right) = 0$
- $(2\pi, 1) \rightarrow f(2\pi) = 1$



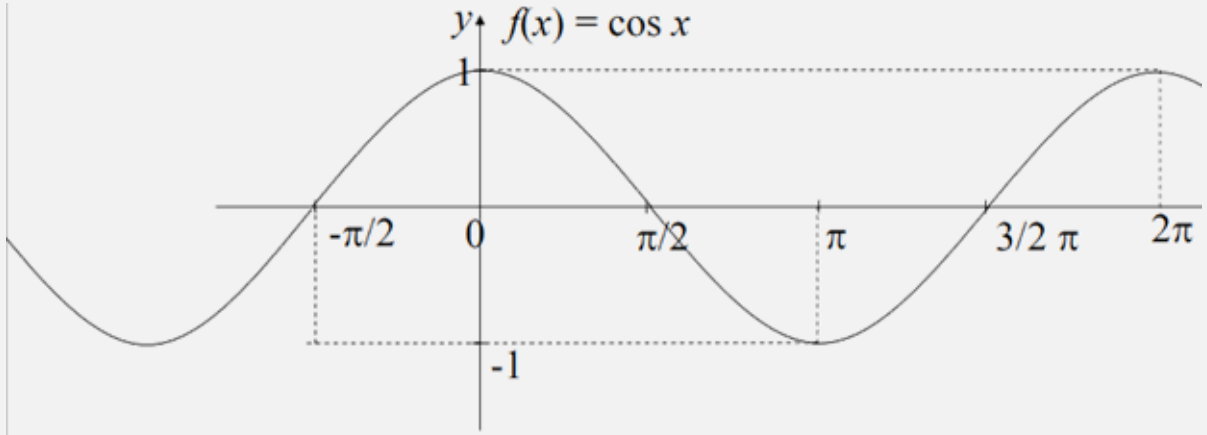
f is bounded
f is strictly increasing in $[\pi + 2k\pi, 2\pi + 2k\pi], k \in \mathbb{Z}$
f is strictly decreasing in $[2k\pi, \pi + 2k\pi], k \in \mathbb{Z}$

Tangent Function

$$f(x) = \tan x = \operatorname{tg} x$$
$$= \frac{\sin x}{\cos x}, \text{with } \cos x \neq 0$$

$$f : x \in \mathbb{R} - \left\{-\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right\}$$

The domain of the tangent function is the whole set $\mathbb{R}$ deprived of the values that make the cosine in the denominator equal to zero.



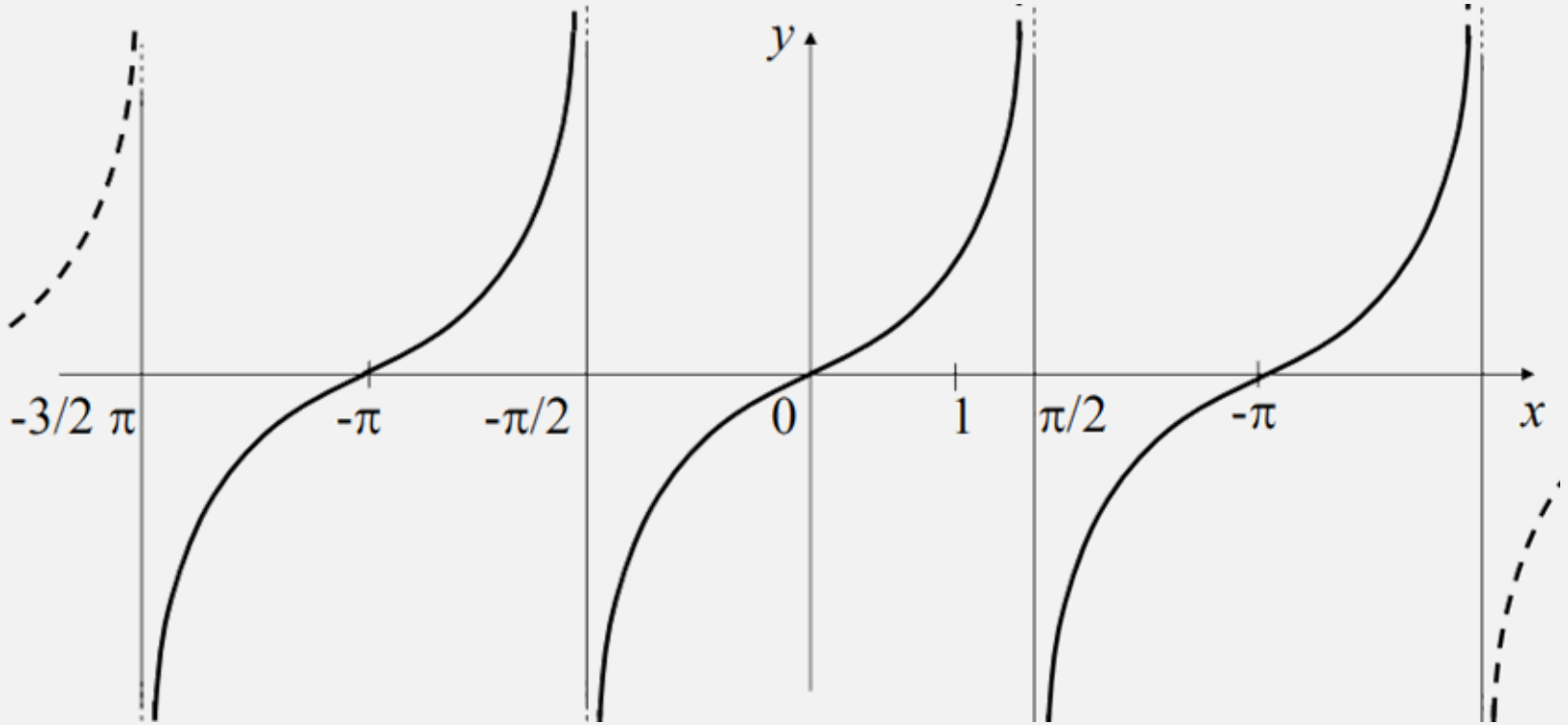
$$\cos x \neq 0 \text{ se } x \neq \ldots, -\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3}{2}\pi, \ldots$$

$$\cos x \neq 0 \text{ se } x \neq \frac{\pi}{2} + k\pi, k \in Z$$

$$f(x) = \tan x = \operatorname{tg} x = \frac{\sin x}{\cos x}, \text{with} \cos x \neq 0$$

$$f : x \in \mathbb{R} - \left\{-\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right\} = f : x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\} \rightarrow \tan x \in \mathbb{R}$$

The **codomain** of the tangent function is $\mathbb{R}$



For each fixed value of the arc x, the tangent function assumes a corresponding numerical value.

x	$\tan x$
0	0
π/4	1
π/2	/
π	0
π/6	$\frac{\sqrt{3}}{3}$
π/3	$\sqrt{3}$
3/2 π	/
2 π	0

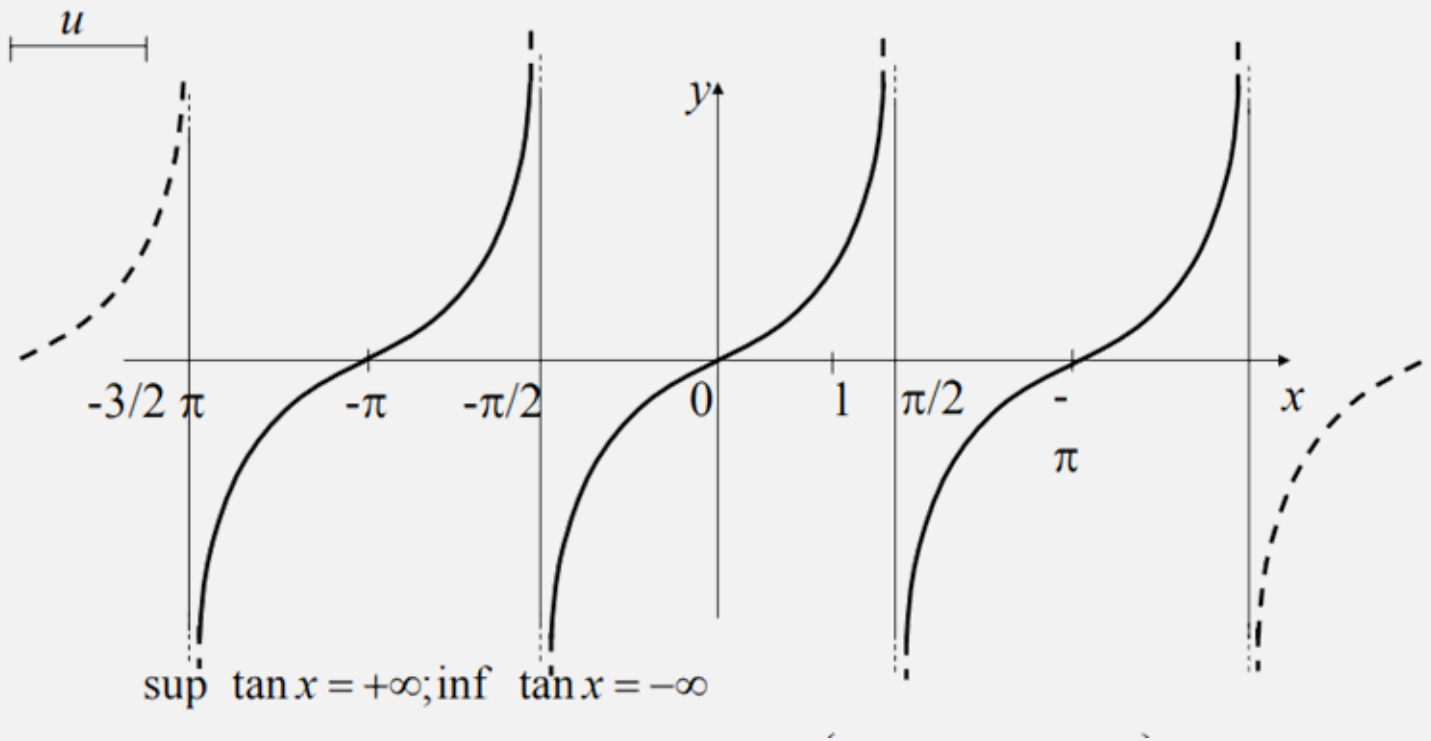
Periodicity
At each half revolution (corresponding to an arc of length π), the tangent function takes the same values.

→ therefore, the tangent function is **periodic** with period π:

$$\tan(x + \pi) = \tan x$$
$$\forall x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$$

Notable points

- $(0,0) \rightarrow f(0) = 0$
- $(\pi, 0) \rightarrow f(\pi) = 0$
- $(2\pi, 0) \rightarrow f(2\pi) = 0$



f is unbounded

f is strictly increasing in $(-\pi/2 + k\pi, \pi/2 + k\pi), \forall k \in \mathbb{Z}$

Since trigonometric functions are periodic, they are not injective.

However, they are strictly increasing or decreasing on suitable intervals: the restrictions to these intervals are **injective** and therefore **invertible**.

For each trigonometric function, a **fundamental interval** is chosen, that is, a set on which the restriction of the function is injective.

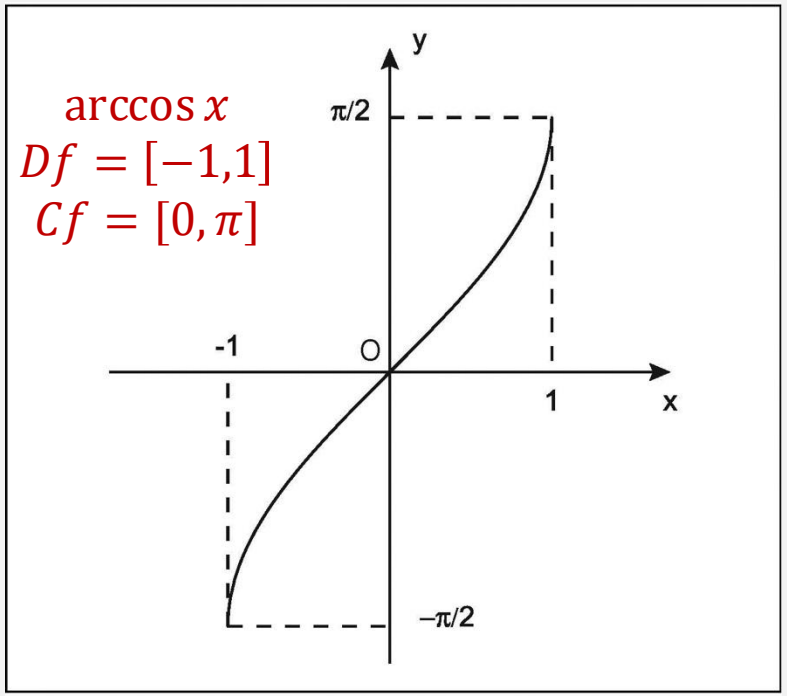
Within each fundamental interval, the inverse function can be defined.

Trigonometric function	*Fundamental interval*	*Inverse function*
$\sin x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\arcsin x$
$\cos x$	$[0, \pi]$	$\arccos x$
$\tan x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\arctan x$

The following identities hold (for all the trigonometric functions):

$\arcsin(\sin x) = x, \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

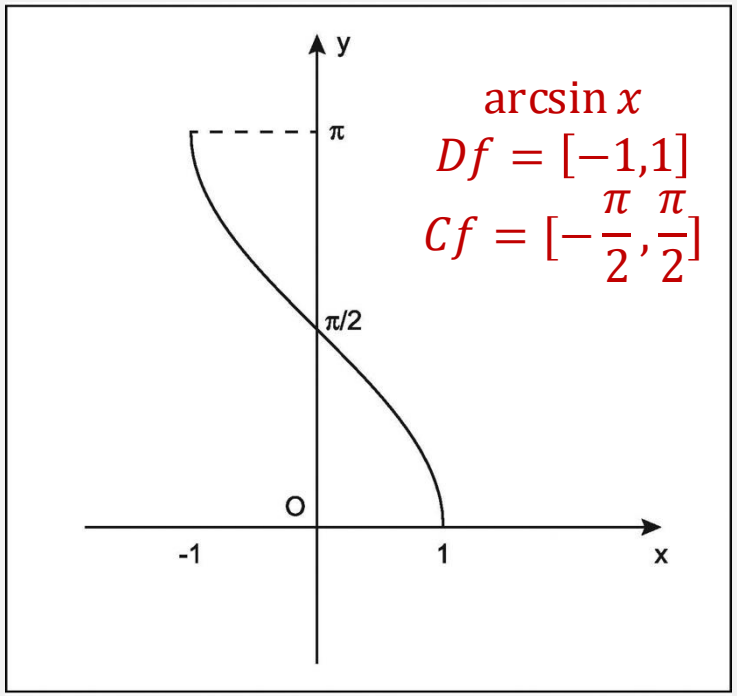
$\sin(\arcsin x) = x, \forall x \in [-1,1]$



$\arccos x$

$Df = [-1,1]$

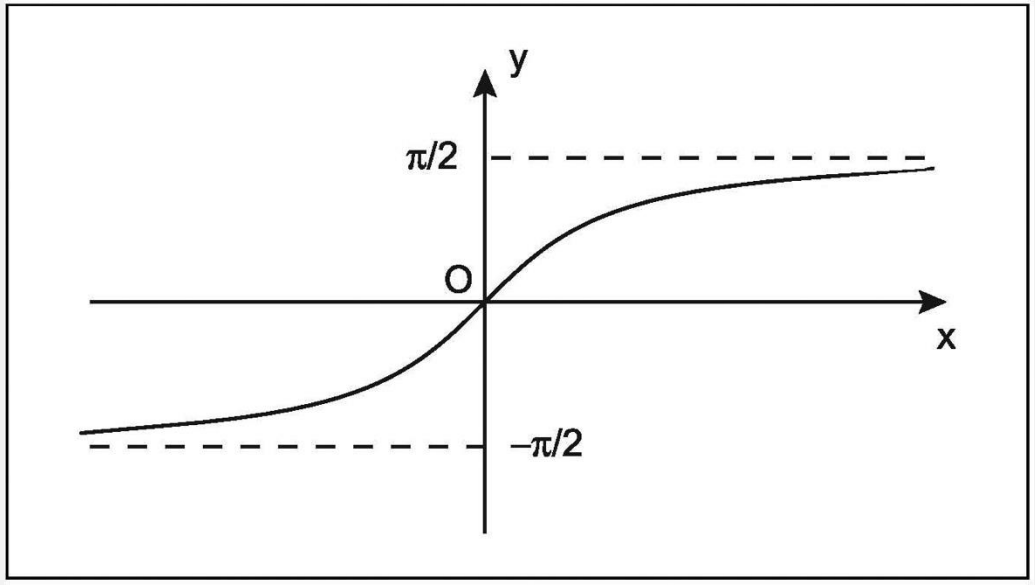
$Cf = [0, \pi]$



$\arcsin x$

$Df = [-1,1]$

$Cf = [-\frac{\pi}{2}, \frac{\pi}{2}]$



$\arctan x$

$Df = \mathbb{R}$

$Cf = (-\frac{\pi}{2}, \frac{\pi}{2})$

Example – Periodic Biosignals

Many biological quantities repeat in time: heartbeat, breathing, hormones, temperature.

A sine or cosine captures a rhythm through its period and amplitude:

$T(t) = T_0 + A \cdot sin(2\pi t/P)$, period P

Body temperature oscillates about 1 °C over a 24-hour (circadian) period.

Period, amplitude and phase are how we quantify any biological rhythm.

