APPLIED PHYSICS AND BIOSTATISTICS

MODULE MATH4 — Limits, Continuity, Asymptotes

Describing the Limiting Behaviour of Biological Systems

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MODULE MATH4 — LIMITS

Limits Describe Where a Biological Process Is Heading

A limit asks: what value does a quantity approach as its input grows large, or nears a point?

Saturation → as substrate rises, an enzyme's rate approaches a ceiling V_{max} (a horizontal asymptote).

Limited growth → a population approaches its carrying capacity K as t → ∞.

Continuity means no sudden jumps; discontinuities flag thresholds or switch-like behaviour.

Asymptotes are the 'limiting levels' we read off every saturation and growth curve.

1. “Informal” Definition of Limit

$$\lim_{x \to x_0} f(x) = L$$

Let $f : D \to \mathbb{R}$ be a function defined for all x in a neighborhood of x_0, except possibly at $x = x_0$

We say that f tends to the limit L as x tends to x_0, if $f(x)$ takes values arbitrarily close to L whenever x is sufficiently close to x_0 (from both sides), excluding the point $x = x_0$.

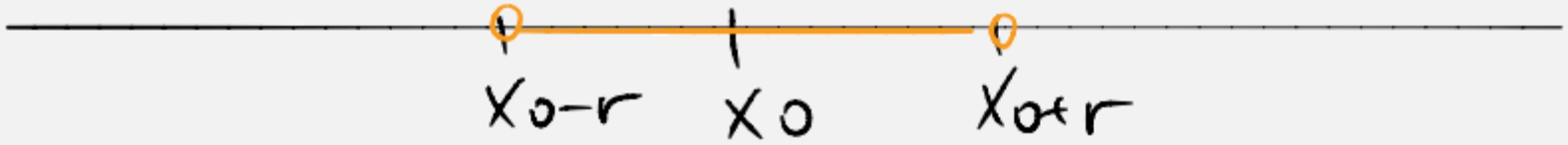
Recall: Definition of Neighborhood

Let $x_0 \in \mathbb{R}$ and $r > 0$.

We consider

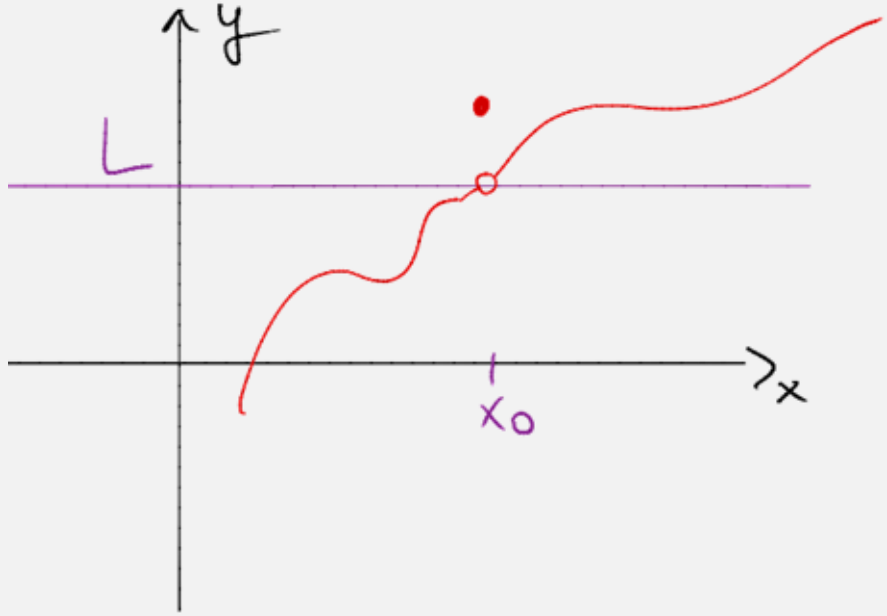
$$I_r(x_0) = \{x \in \mathbb{R} : |x - x_0| < r\} = \{x \in \mathbb{R} : x_0 - r < x < x_0 + r\},$$

the open neighborhood of center x_0, that is, the set of real numbers whose distance from x_0 is less than $r > 0$.



2. Definition via Neighborhoods

$$\lim_{x \to x_0} f(x) = L$$



with

- $L \in \mathbb{R}$

We say that the function f tends to the limit L as $x \to x_0$ if:

For every neighborhood $I(L, \varepsilon)$, with $\varepsilon > 0$, there exists a neighborhood $I(x_0, \delta)$, with $\delta > 0$, such that for all $x \in (I(x_0, \delta) - \{x_0\}) \cap D$ it follows that $f(x) \in I(L, \varepsilon)$

$$\forall I(L, \varepsilon), \varepsilon > 0 \; \exists \, I(x_0, \delta), \delta > 0 : \forall x \in (I(x_0, \delta) - \{x_0\}) \cap D \Longrightarrow f(x) \in I(L, \varepsilon)$$

Cases of Limits

General case:

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall I_L \; \exists \; I_{x0} : \; \forall x \in I_{x0} - \{x_0\} \Longrightarrow f(x) \in I_L$$

if and only if for every neighborhood of L there exists a neighborhood of x_0 such that, for all x belonging to the neighborhood of x_0 (possibly excluding x_0), $f(x)$ belongs to the neighborhood of L.

The cases may involve:

$$L = \begin{cases} \in \mathbb{R} \to I_L = (L - \varepsilon, L + \varepsilon) \\ +\infty \to I_L = (n, +\infty) \\ -\infty \to I_L = (-\infty, -n) \end{cases}$$

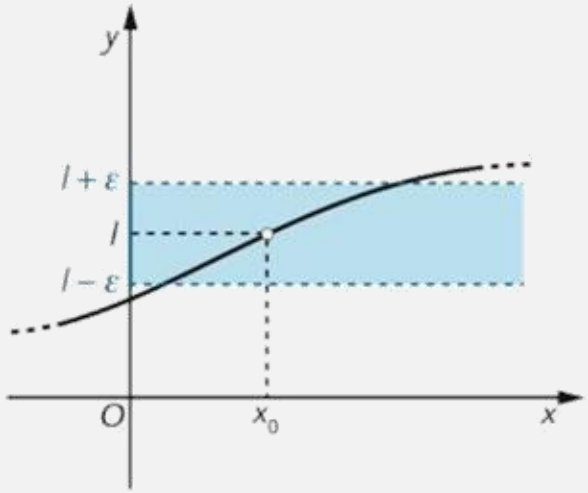
$$x_0 = \begin{cases} \in \mathbb{R} \to I_L = (L - \varepsilon, L + \varepsilon) \\ +\infty \to I_L = (n, +\infty) \\ -\infty \to I_L = (-\infty, -n) \end{cases}$$

- Finite limit at a finite point: $L, x_0 \in \mathbb{R}$ $\lim_{x \to x_0} f(x) = L$

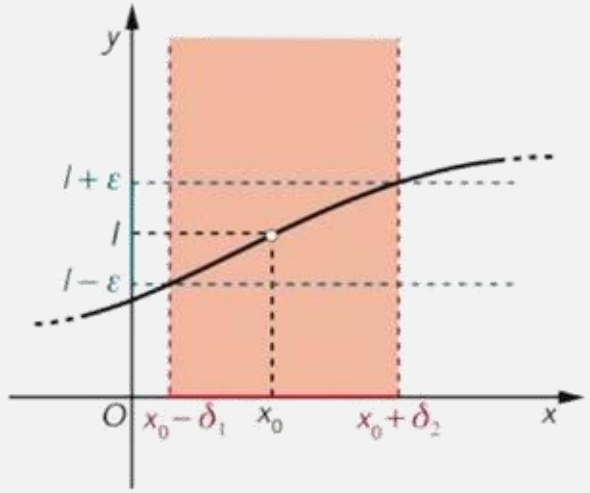
Let $f(x)$ be a function defined in an interval (a, b) excluding the point x_0.

We say that $f(x)$ tends to the limit L as x tends to x_0 if, for every arbitrary positive number ε, one can determine a neighborhood of x_0 such that for every x in that neighborhood:

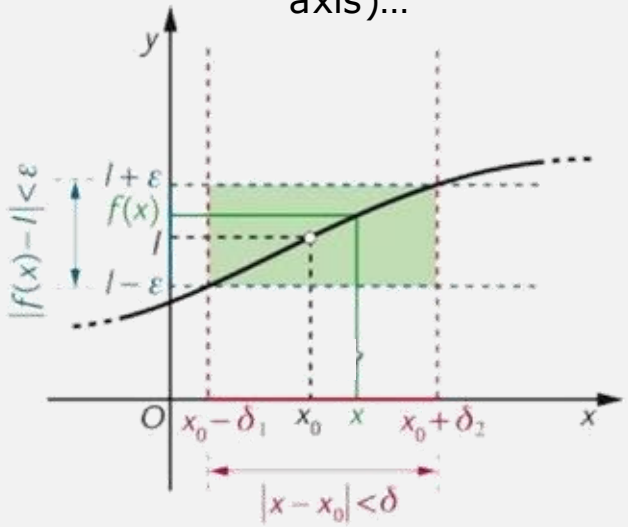
$$|f(x) - L| < \varepsilon \to L - \varepsilon < f(x) < L + \varepsilon$$



For every $\varepsilon > 0$ (by choosing ε, we select an arbitrary neighborhood of l on the y-axis)…



… there exists $\delta > 0$ (that identifies a proper neighborhood of x_0 on the x-axis)…



such that for each $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$ we have:
$f(x) \in (l - \varepsilon, l + \varepsilon)$

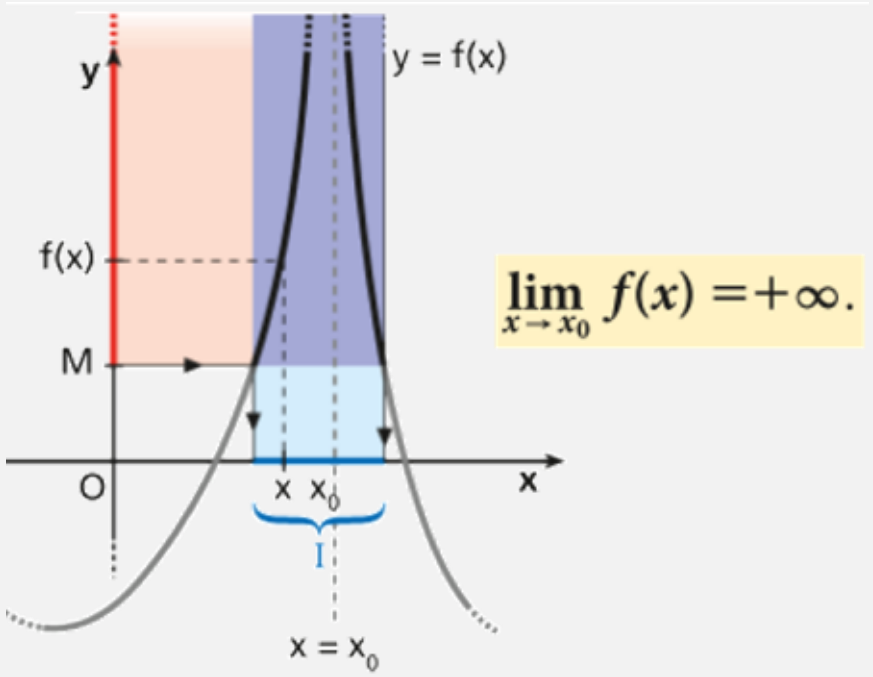
$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \; \exists \, I_{x0} : \; \forall x \in I_{x0} - \{x_0\} \Longrightarrow |f(x) - L| < \varepsilon$$

Infinite limit at a finite point:

$$L = \infty, x_0 \in \mathbb{R} \quad \lim_{x \to x_0} f(x) = \infty$$

Let $f(x)$ be a function defined on an interval (a, b), excluding the point x_0.
We say that the function $f(x)$ tends to infinity as x approaches x_0 if, for every arbitrary positive number M, it is always possible to determine a (punctured) neighborhood of the point x_0 such that, for every x in that neighborhood, the following inequality holds:

$$|f(x)| > M$$



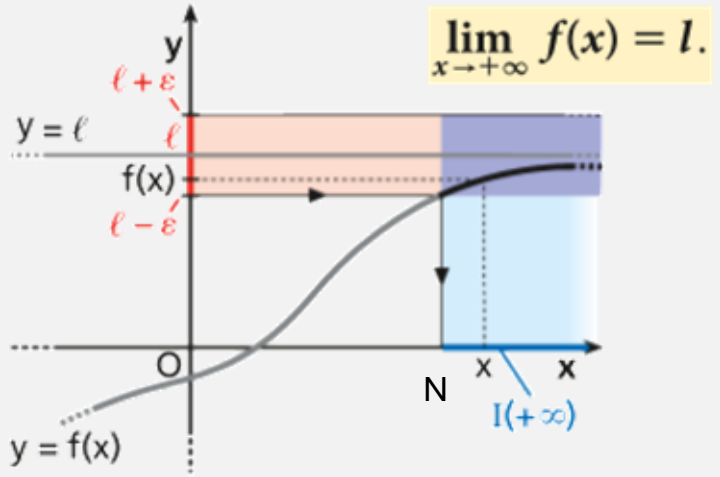
$$\lim_{x \to x_0} f(x) = \infty \Leftrightarrow \forall n > 0 \; \exists \, I_{x0} : \; \forall x \in I_{x0} - \{x_0\} \Longrightarrow |f(x) - L| > M$$

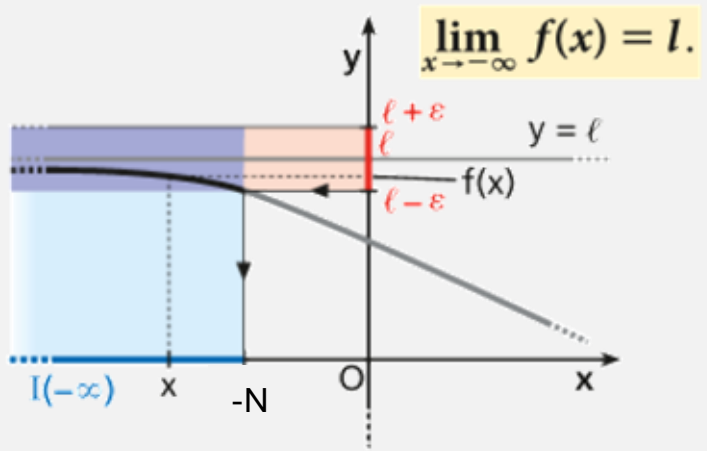
Finite limit at infinity:

$$L \in \mathbb{R}, x_0 = \infty \quad \lim_{x \to \infty} f(x) = L$$

Let $f(x)$ be a function defined on an interval (a, b).
We say that $f(x)$ tends to the limit L as x tends to infinity if, for every arbitrary positive number ε, it is always possible to determine a number $N > 0$ such that, for every x satisfying $|x| > N$, the following condition holds:

$$|f(x) - L| < \varepsilon \to L - \varepsilon < f(x) < L + \varepsilon$$





$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \; \exists \, N > 0 : \; \forall x : |x| > N \Longrightarrow |f(x) - L| < \varepsilon$$

Right-hand and Left-hand Limits (Extended Real Line)

- Let $x_0 = +\infty$: we call open neighborhood of x_0 an open half-line $(a, +\infty)$ and close neighborhood $[a, +\infty)$
- Let $x_0 = -\infty$: we call open neighborhood of x_0 an open half-line $(-\infty, a)$ and close neighborhood $(-\infty, a]$
- Let $a \subset \mathbb{R}$, $+\infty$ is an accumulation point of A if

$$A \cap (a, +\infty) \neq \emptyset \quad \forall a > 0$$

- Let $a \subset \mathbb{R}$, $\pm\infty$ is an accumulation point of A if

$$A \cap (-\infty, a) \neq \emptyset \quad \forall a < 0$$

Unified Definition of Limit

Let $x_0, L \in \overline{\mathbb{R}}$.
Let $f : D_f \to \mathbb{R}$.
Let x_0 accumulation point for D_f.
$f(x)$ tends to L as x tends to x_0 → $\lim_{x \to x_0} f(x) = L$, if, for every neighborhood J of L there exists a neighborhood I of x_0 such that:

$$\forall x \in \left(I \cap D_f\right) - \{x_0\} \text{ one has that: } \quad f(x) \in J$$

Right-hand Limit

Let $x_0 \in \mathbb{R}$ and let $f : D_f \to \mathbb{R}$, assume that x_0 is an accumulation point of the set $D_f \cap (x_0, +\infty)$.

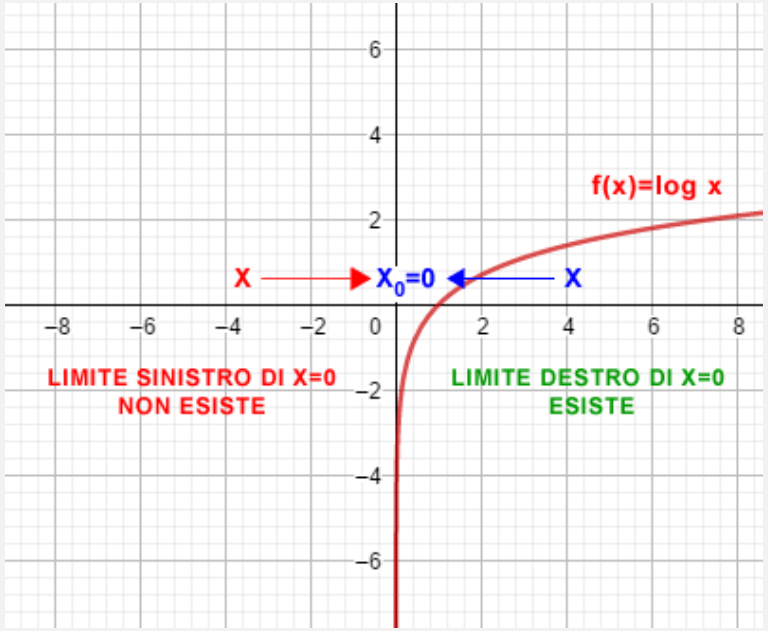
If the limit of the restriction of f to $D_f \cap (x_0, +\infty)$ exists as $x \to x_0$, then this value is called the right-hand limit of f at x_0, denoted by: :

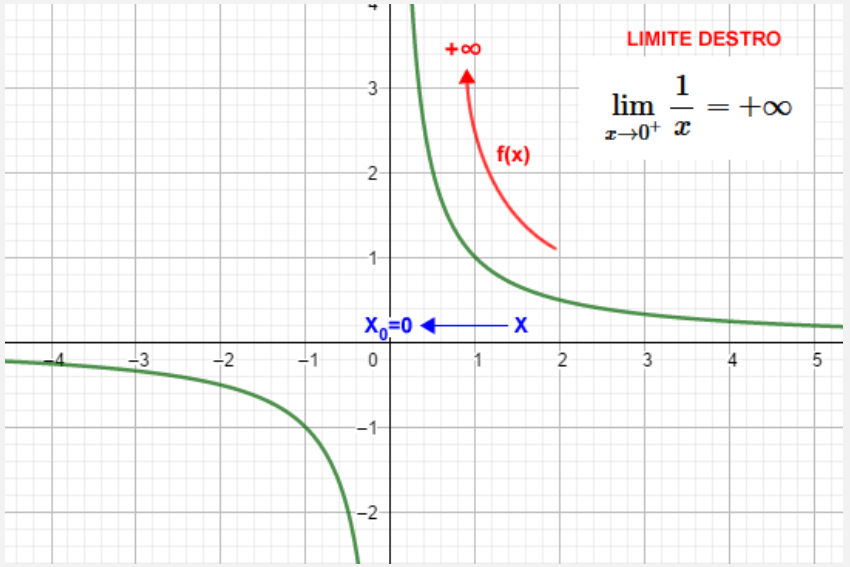
$$\lim_{x \to x_0^+} f(x)$$

Quantifier formulation (for $L \in \mathbb{R}$):

$$\lim_{x \to x_0^+} f(x) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in (x_0, x_0 + \delta) \cap D_f \; \to \quad |f(x) - L| \leq \varepsilon$$

$$x_0 < x < x_0 + \delta$$



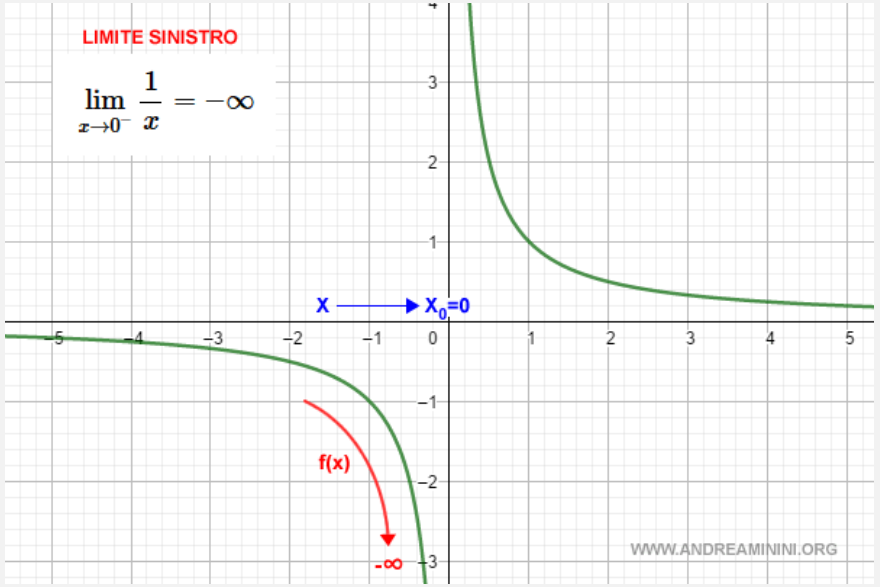


Left-hand Limit

Let $x_0 \in \mathbb{R}$ and let $f : D_f \to \mathbb{R}$, assume that x_0 is an accumulation point of the set $D_f \cap (-\infty, x_0)$.

If the limit of the restriction of f to $D_f \cap (-\infty, x_0)$ exists as $x \to x_0$, then this value is called the left-hand limit of f at x_0, denoted by:

$$\lim_{x \to x_0^-} f(x)$$



Quantifier formulation (for $L \in \mathbb{R}$):

$$\lim_{x \to x_0^-} f(x) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in (x_0, x_0 + \delta) \cap D_f \; \to \quad |f(x) - L| \leq \varepsilon$$

$$x_0 - \delta < x < x_0$$

Limits of Polynomial Functions

From the limit laws, it follows that the limit as $x \to x_0 \in \mathbb{R}$ exists for every polynomial function:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Using the definition of limit, it is easy to verify that:

$$\lim_{x \to x_0} a_0 = a_0, \qquad \lim_{x \to x_0} a_1 x = a_1 x_0, \qquad \lim_{x \to x_0} a_n x^n = a_n x_0$$

Therefore, in general:

$$\lim_{x \to x_0} p(x) = p(x_0) = a_0 + a_1 x_0 + \cdots + a_n x_0^n$$

Limits of Quotients of Polynomials (Rational Functions)

$$\lim_{x\to\pm\infty}\frac{P_n(x)}{Q_m(x)} = \begin{matrix} 1.\, n > m \to \lim = \pm\infty \\ 2.\, n < m \to \lim = 0 \\ 3.\, n = m \to \lim = a_n/b_m \end{matrix}$$

Examples.

1. $\lim_{x\to-\infty}\frac{1-x^2+3x^3}{2x^2-1+5x^3} = \frac{3}{5} \to n = m$

2. $\lim_{x\to-\infty}\frac{5x^3+2}{12x^4-x+7} = 0^- \to n < m$

3. $\lim_{x\to1^{\pm}}\frac{x^3-7x+1}{x^2-1} = \frac{-3}{0} = \pm\infty \to n > m$

Indeterminate Forms

Indeterminate forms arise from operations involving infinitesimals and infinities for which the limit cannot be determined a priori:

$$\infty - \infty \quad 0 \cdot \infty \quad \frac{\infty}{\infty} \quad \frac{0}{0} \quad 0^0 \quad 1^\infty \quad \infty^0$$

Non-indeterminate forms include:

$$k + \infty = +\infty$$
$$k - \infty = -\infty$$
$$\infty + \infty = \infty$$
$$k \cdot \infty = \infty$$
$$\infty \cdot \infty = \infty$$
$$\frac{k}{\infty} = 0$$
$$\frac{0}{\infty} = 0$$
$$\frac{\infty}{k} = \infty$$
$$\frac{k}{0} = \infty$$

Examples

$$\lim_{x\to\pm\infty} \frac{polinomio}{polinomio} = \frac{\pm\infty}{\pm\infty}$$

the numerator has a higher degree

$$\lim_{x\to+\infty} \frac{4x^5 - 3x + 2}{2x^2 + 4} = +\infty \qquad \lim_{x\to+\infty} \frac{-3x^2 + 2x - 5}{x - 4} = -\infty$$

$$\lim_{x\to-\infty} \frac{4x^5 - 3x + 2}{2x^2 + 4} = -\infty \qquad \lim_{x\to-\infty} \frac{2x^4 + 5x^3}{x^3 + 2x^2 - 1} = +\infty$$

the degrees are equal

$$\lim_{x\to+\infty} \frac{7x^3 - 3x + 2}{2x^3 + 4} = \frac{7}{2} \qquad \lim_{x\to+\infty} \frac{3x^2 - x + 2}{4x - 2x^2 - 1} = -\frac{3}{2}$$

$$\lim_{x\to-\infty} \frac{7x^3 - 3x + 2}{2x^3 + 4} = \frac{7}{2} \qquad \lim_{x\to-\infty} \frac{3x^2 - x + 2}{4x - 2x^2 - 1} = -\frac{3}{2}$$

the denominator has a higher degree

$$\lim_{x\to+\infty} \frac{5x^3 - 3x + 1}{2x^4 + 4} = 0 \qquad \lim_{x\to+\infty} \frac{x^2 + 2x + 1}{2x^3 + 4} = 0$$

$$\lim_{x\to-\infty} \frac{5x^3 - 3x + 1}{2x^4 + 4} = 0 \qquad \lim_{x\to-\infty} \frac{2x + 1}{-2x^2 + 4x + 3} = 0$$

Limits of Continuous Functions

Let a real-valued function f defined on an interval I be continuous at $x_0 \in I$

f is continuous at x_0 if:

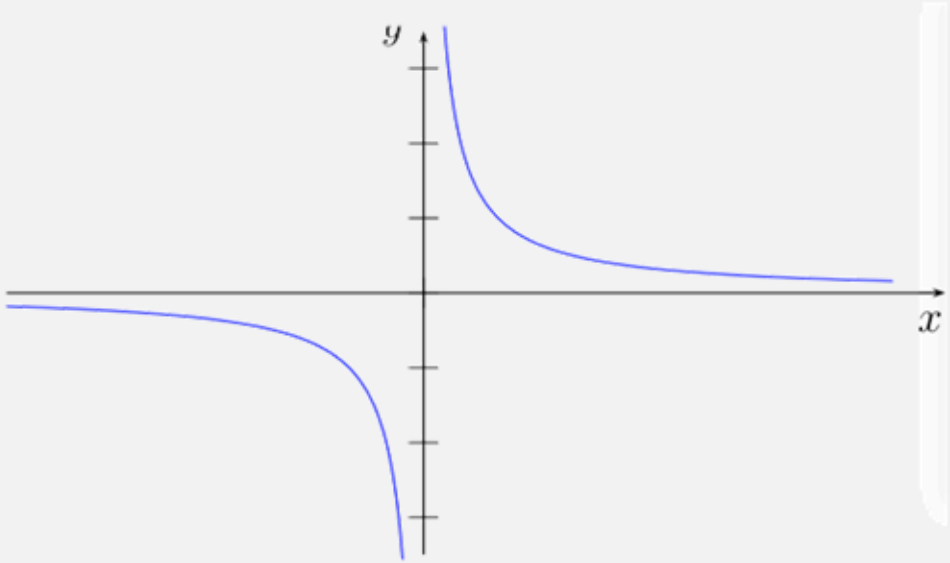
$$\lim_{x \to x_0} f(x) = f(x_0)$$

Hence, f is continuous in a point x_0 internal to the definition interval if:

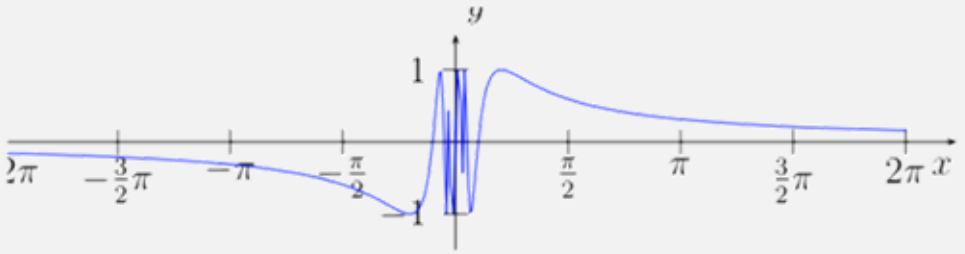
- $\exists f(x_0)$
- $\exists$ finite limit of $f(x)$ as $x \to x_0$
- the limit coincides with the value $f(x_0)$ of f at x_0

Limits of Continuous Functions

$f : x \mapsto \frac{1}{x}$ is continuous at $D_f = \mathbb{R} \setminus \{0\}$

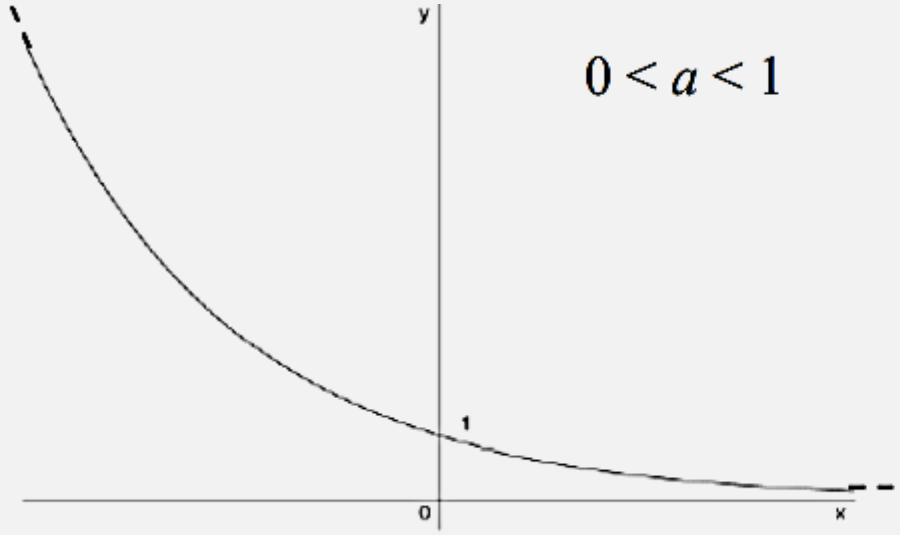


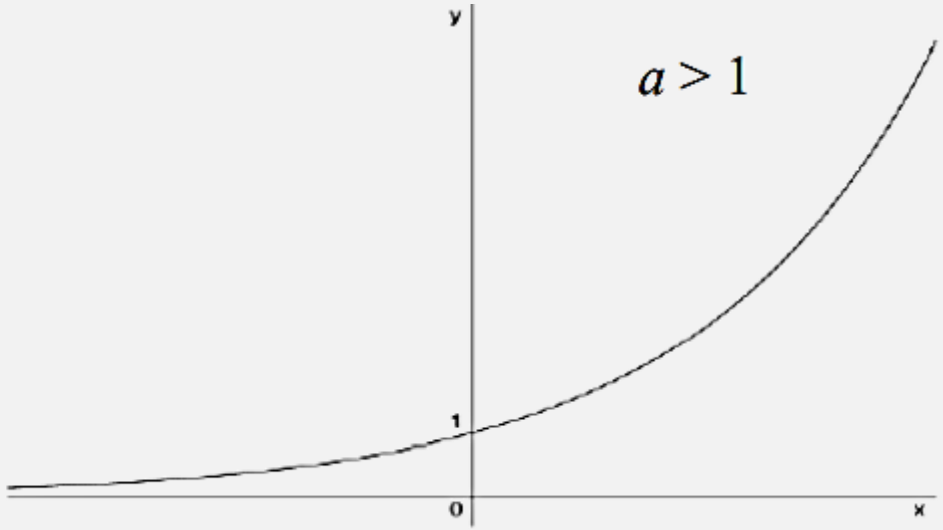
$f : x \mapsto \sin\left(\frac{1}{x}\right)$ is continuous at $\operatorname{dom} f = \mathbb{R} \setminus \{0\}$



Limits of Elementary Functions

Exponential function





$$\lim_{x \to +\infty} a^x = 0$$

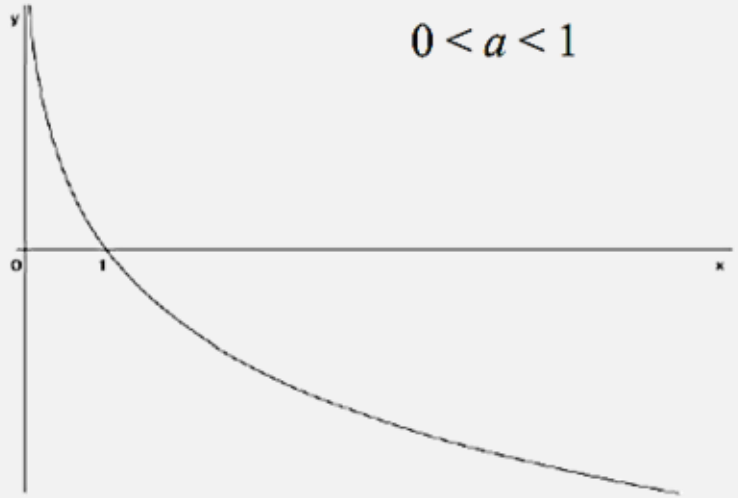
$$\lim_{x \to +\infty} a^x = +\infty$$

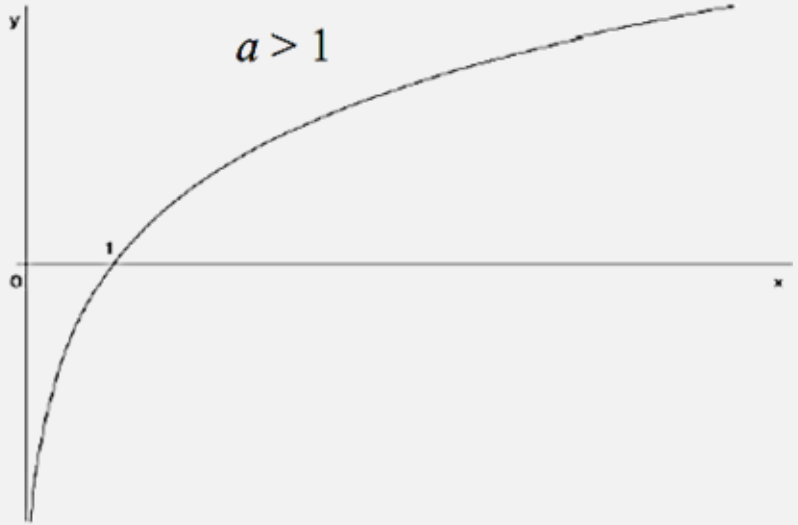
$$\lim_{x \to -\infty} a^x = +\infty$$

$$\lim_{x \to -\infty} a^x = 0$$

Limits of Elementary Functions

Logarithmic function





$$\lim_{x \to +\infty} \log_a x = -\infty$$

$$\lim_{x \to +\infty} \log_a x = +\infty$$

$$\lim_{x \to 0^+} \log_a x = +\infty$$

$$\lim_{x \to 0^+} \log_a x = -\infty$$

Horizontal Asymptotes

Given a real-valued function f defined on an unbounded interval I, if the limit at infinity is finite, that is,:

$$\lim_{x \to \pm\infty} f(x) = l$$

then, the line $y = l$ is called horizontal asymptote for the graph of $f(x)$

$$\lim_{x \to -\infty} f(x) = 2$$

$$\lim_{x \to +\infty} f(x) = 2$$

$$y = 2$$

horizontal asymptote



Horizontal Asymptotes

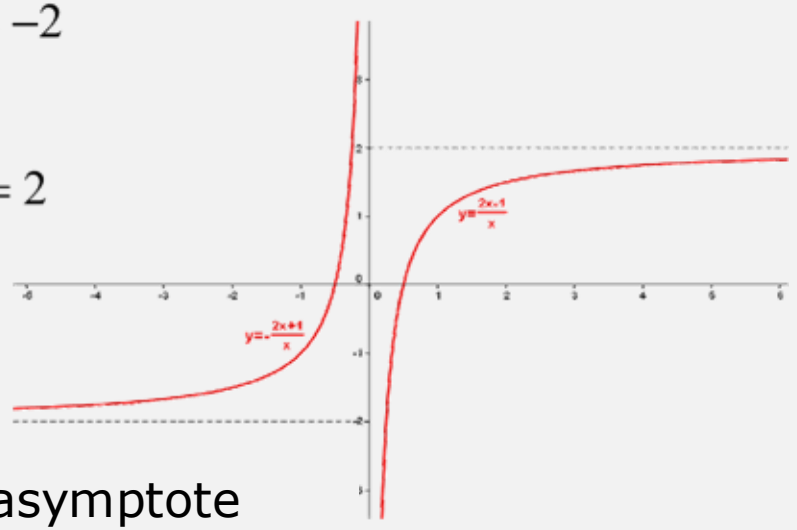
If

$$\lim_{x\to+\infty} f(x) = l_1 \qquad \lim_{x\to-\infty} f(x) = l_2$$

then the graph of $f(x)$ admits two horizontal asymptotes: $y = l_1$ as $x \to +\infty$ and $y = l_2$ as $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = -2$$

$$\lim_{x\to+\infty} f(x) = 2$$



$y = 2$

right horizontal asymptote

$y = -2$

left horizontal asymptote

Example - Enzyme Saturation and V_{max}

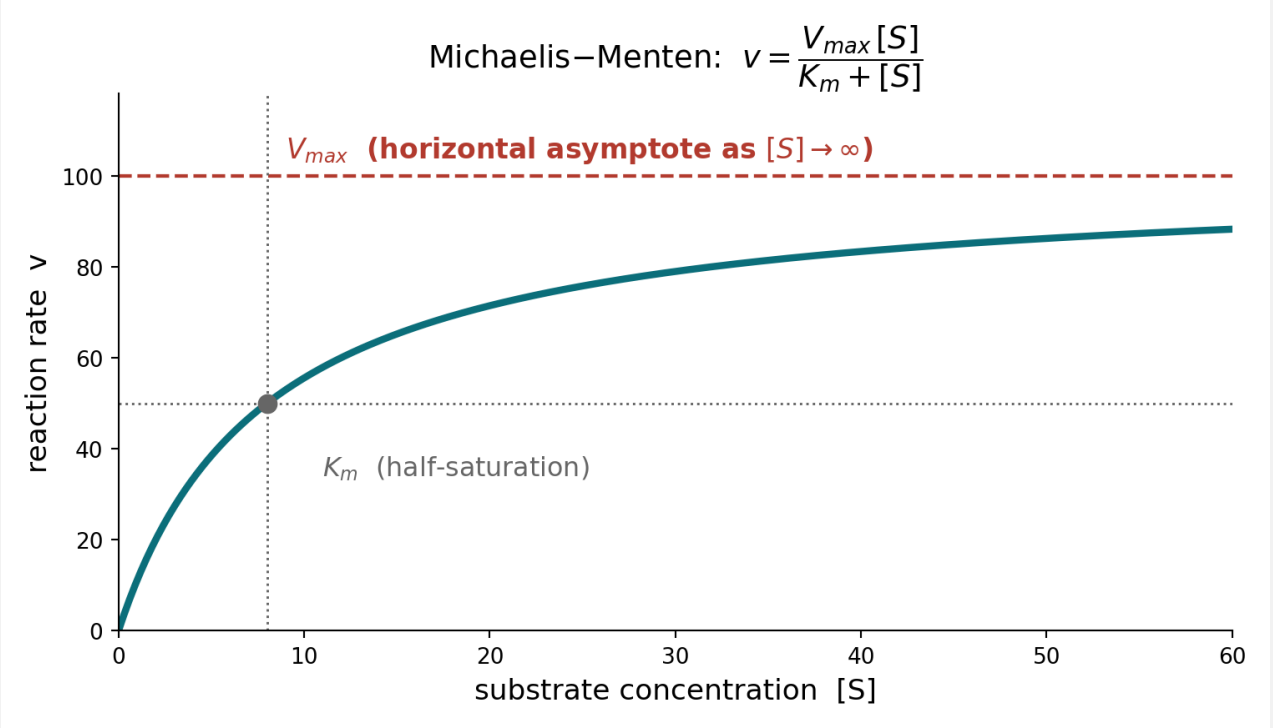
As substrate concentration [S] increases, the reaction rate does not grow without bound.

It approaches a maximum rate V_{max} — a horizontal asymptote of the curve:

$$\lim_{[S]\to+\infty} v = V_{max}$$

$$v = V_{max} \cdot [S]/(K_m + [S])$$

K_m, the substrate level giving half of V_{max}, is the enzyme's affinity marker.



Oblique Asymptotes

Given a real-valued function f defined on an unbounded interval I, if

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

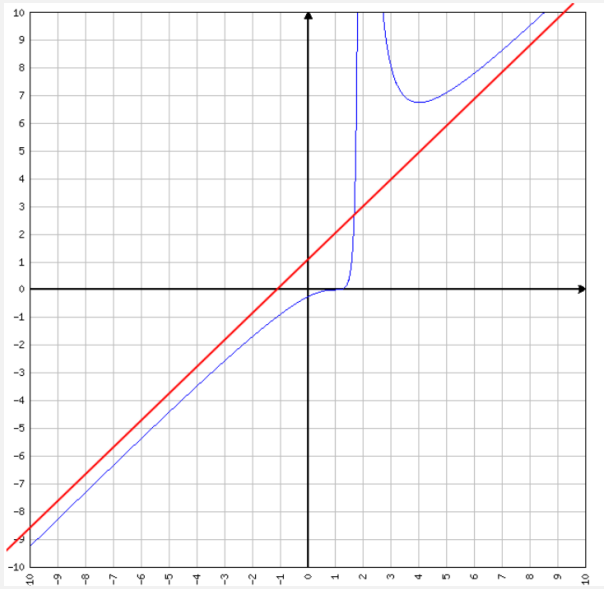
then the function DOES NOT admit a horizontal asymptote.
It is possible to check whether the graph admits an oblique asymptote of equation $y = mx + q$.

The graph of $f(x)$ admits an oblique asymptote if:

$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = m \neq 0 \; finite$$

$$\lim_{x\to\pm\infty} [f(x) - mx] = q \; finite$$



Vertical Asymptotes

Given a real-valued function f defined on an interval I missing at most the point x_0 , if the limit at the finite point x_0 is infinite, that is,

$$\lim_{x \to x_0^+} f(x) = \pm\infty$$

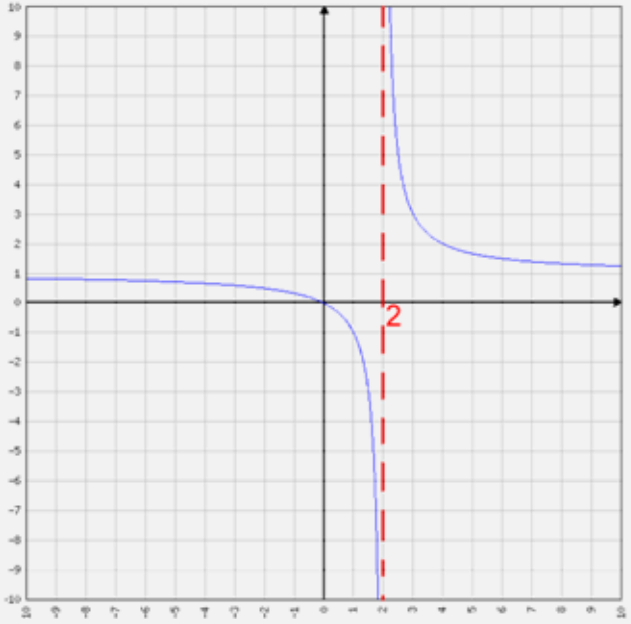
then the line $x = x_0$ is called a vertical asymptote of the graph of $f(x)$

$$\lim_{x \to 2^+} f(x) = +\infty$$

$$\lim_{x \to 2^-} f(x) = -\infty$$

$x = 2$

vertical asymptote

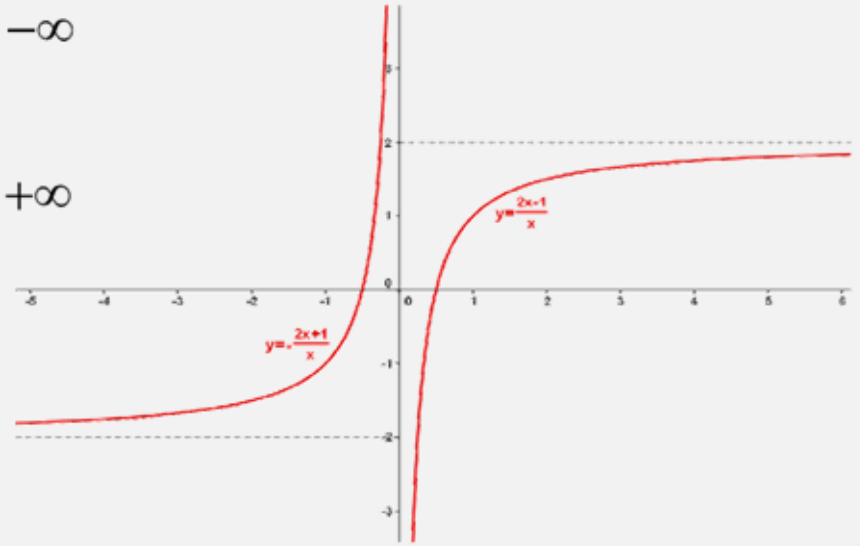


$$\lim_{x \to 0^+} f(x) = -\infty$$

$$\lim_{x \to 0^-} f(x) = +\infty$$

$x = 0$

vertical asymptote



Example.

Determine the possible asymptotes of the following function:: $f(x) = \frac{x^2-4}{x+1}$

The first step in determining the presence of asymptotes is to determine the domain of the function:

$$x - 1 \neq 0 \Rightarrow x \neq -1$$

Therefore, the domain is:

$$(-\infty, -1) \cup (-1, +\infty)$$

Since the function is not defined at a point, it makes sense to search for vertical and horizontal asymptotes:

$$\lim_{x \to -1^-} \frac{x^2-4}{x+1} = \frac{-3}{-1^- +1} = \frac{-3}{0^-} = +\infty$$

$$\lim_{x \to -1^+} \frac{x^2-4}{x+1} = \frac{-3}{-1^+ +1} = \frac{-3}{0^+} = -\infty$$

$\Rightarrow x = -1$ vertical as.

$$\lim_{x \to -\infty} \frac{x^2-4}{x+1} = \left[\frac{\infty}{\infty}\right]$$

$$\Rightarrow \lim_{x \to -\infty} \frac{x^2-4}{x+1} = \lim_{x \to -\infty} \frac{x^2}{x} = \lim_{x \to -\infty} x = -\infty$$

$\nexists$ left horiz. as.

$$\lim_{x \to +\infty} \frac{x^2-4}{x+1} = \left[\frac{\infty}{\infty}\right]$$

$$\Rightarrow \lim_{x \to +\infty} \frac{x^2-4}{x+1} = \lim_{x \to +\infty} \frac{x^2}{x} = \lim_{x \to +\infty}$$

$\nexists$ right horiz. as.

Discontinuities

- if f is defined on an interval $[a, b]$ excluding at most the point x_0

or

- if f is not continuous at a point x_0,

then x_0 is called a singular point or a point of discontinuity.

First-Kind (Jump) Discontinuities

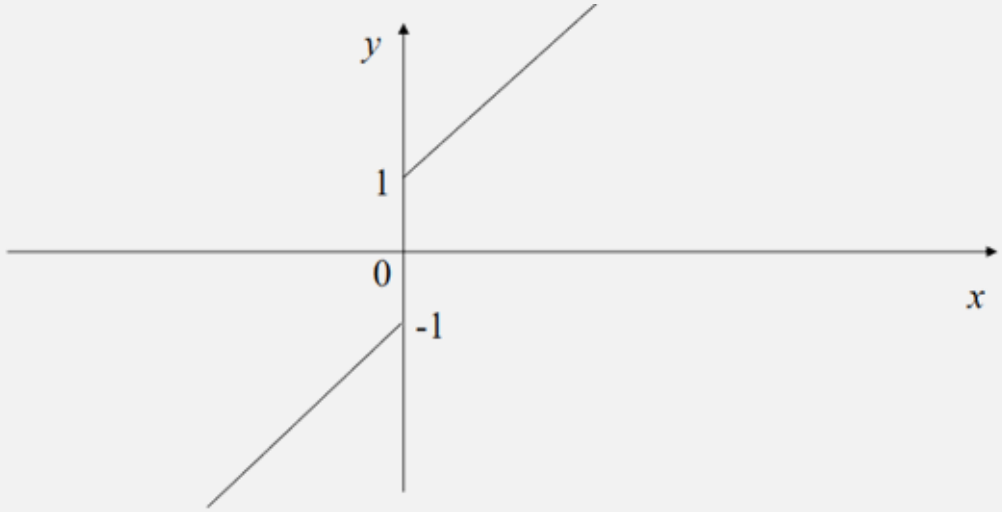
A function f has a first-kind discontinuity at x_0 if both the right-hand and left-hand limits exist and are finite, but are different:

$$l_1 = \lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x) = l_2$$

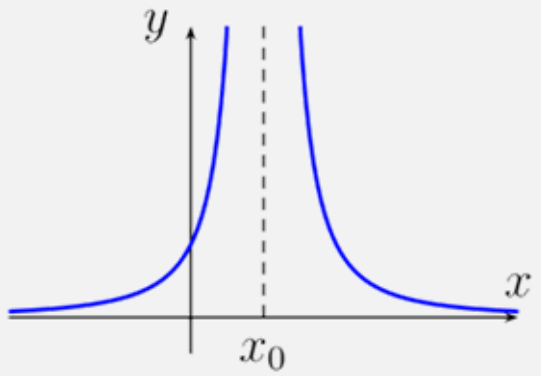
Example. The following function has a first-kind discontinuity at $x = 0$:

$$f(x) = x + \frac{|x|}{x} = \begin{cases} x + 1, se\ x > 0 \\ x - 1, se\ x < 0 \end{cases}$$

$$1 = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) = -1$$



Second-Kind Discontinuities

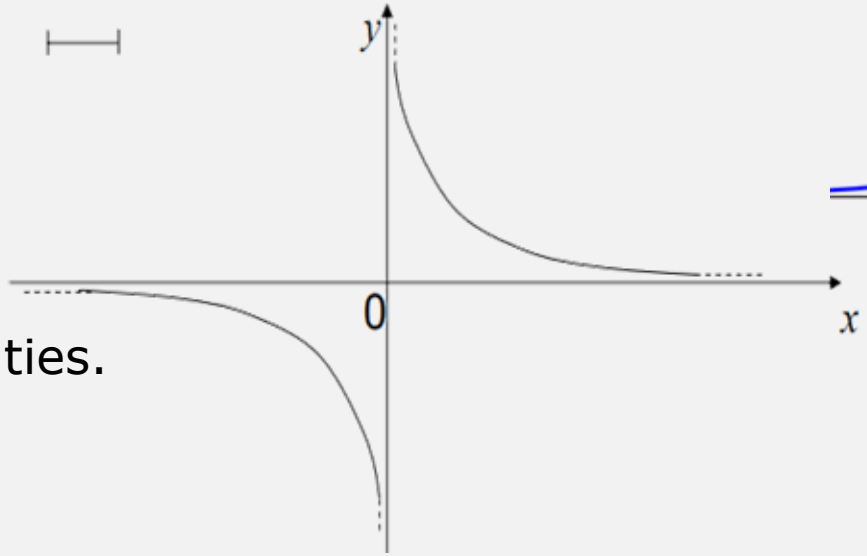


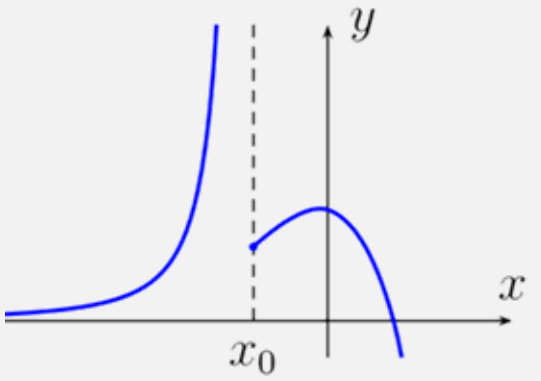
A function f has a second-kind discontinuity at x_0 if at least one of the one-sided limits is infinite or does not exist:

Example. The function has a second-kind discontinuity at $x = 0$:

$$f(x) = \frac{1}{x}$$

$$+\infty = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x) = -\infty$$





The presence of vertical asymptotes is representative of second-kind discontinuities.

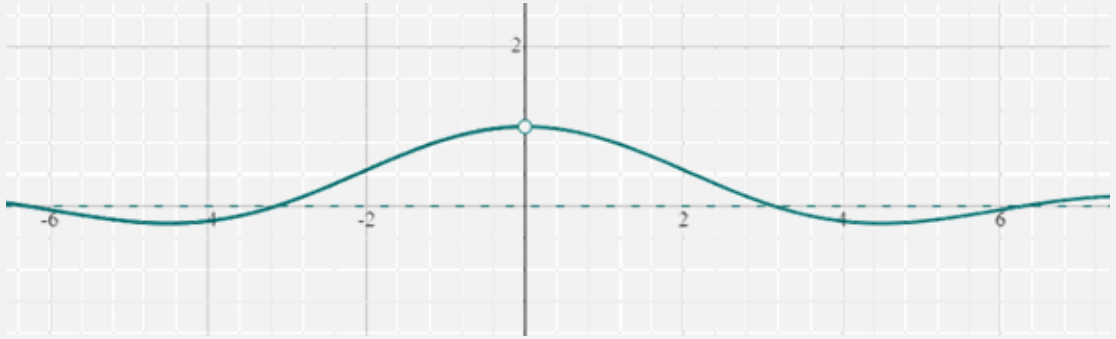
Third-Kind (Removable) Discontinuities

A function f has a removable discontinuity at x_0 if the limit of f at x_0 exists and is finite, but:

- f is not defined at x_0
- $\lim_{x \to x_0} f(x) \neq f(x_0)$

Example. The function has a removable discontinuity at $x = 0$:

$$f(x) = \frac{\sin x}{x}$$



Indeed:

- f is not defined at $x = 0 \Rightarrow \nexists f(0)$
- but there exists $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Remarkable Limits

Limit	Indeterminate form
$\lim_{x\to 0}\frac{\sin x}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$	$\frac{0}{0}$
$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$	1^∞
$\lim_{x\to 0^+} x\cdot\ln x=0$	$0\cdot\infty$
$\lim_{x\to 0}\frac{e^x-1}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\arcsin x}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\arctan x}{x}=1$	$\frac{0}{0}$

Limit	Indeterminate form
$\lim_{x\to 0}\frac{\tan x}{x}=1$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{1-\cos x}{x}=0$	$\frac{0}{0}$
$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$	1^∞
$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e$	$\frac{0}{0}$
$\lim_{x\to 0}\frac{(1+x)^k-1}{x}=k$	$\frac{0}{0}$

Example – Growth Toward a Carrying Capacity

A population (cells, bacteria, a tumour) cannot grow exponentially forever.

Limited resources make N(t) level off at the carrying capacity K:

$$\lim_{t\to\infty} N(t) = K$$

$$N(t) = K/\left(1 + c \cdot e^{(-rt)}\right)$$

The carrying capacity K is the horizontal asymptote of the growth curve.

