APPLIED PHYSICS AND BIOSTATISTICS

MODULE MATH5 — Derivatives: Rates of Change in Living Systems

Instantaneous Rates, Differentiation Rules, Extrema and Function Study

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The Derivative Is a Rate of Change

The derivative measures how fast a quantity changes — its instantaneous rate.

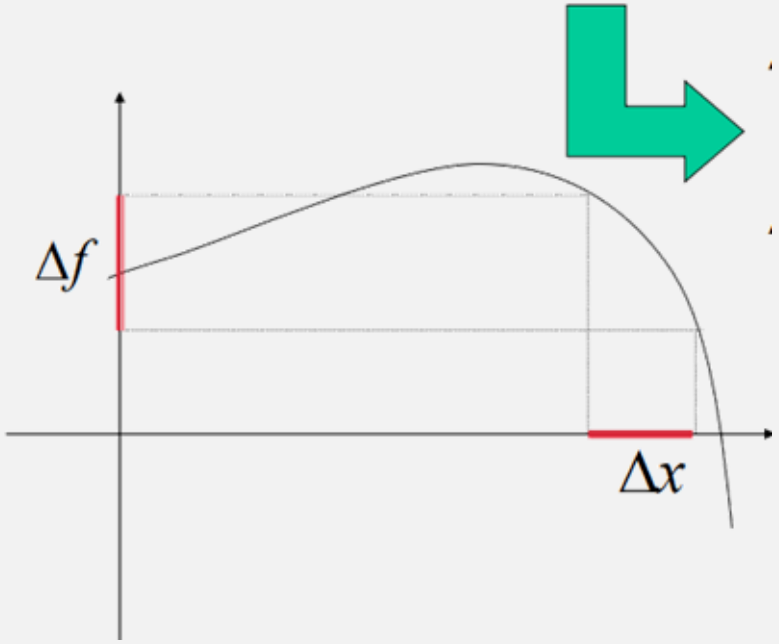
In biology it is a reaction rate, a growth rate, a drug clearance, a velocity.

Where the derivative is zero, a quantity peaks or bottoms out (e.g. the peak drug level).

The sign of the derivative gives increasing vs decreasing; the second derivative gives concavity.
The inflection point (second derivative zero) is where a process grows fastest.

The Difference Quotient

the increment of the variable x is indicated by the symbol Δx
the increment of the function f is indicated by the symbol Δf



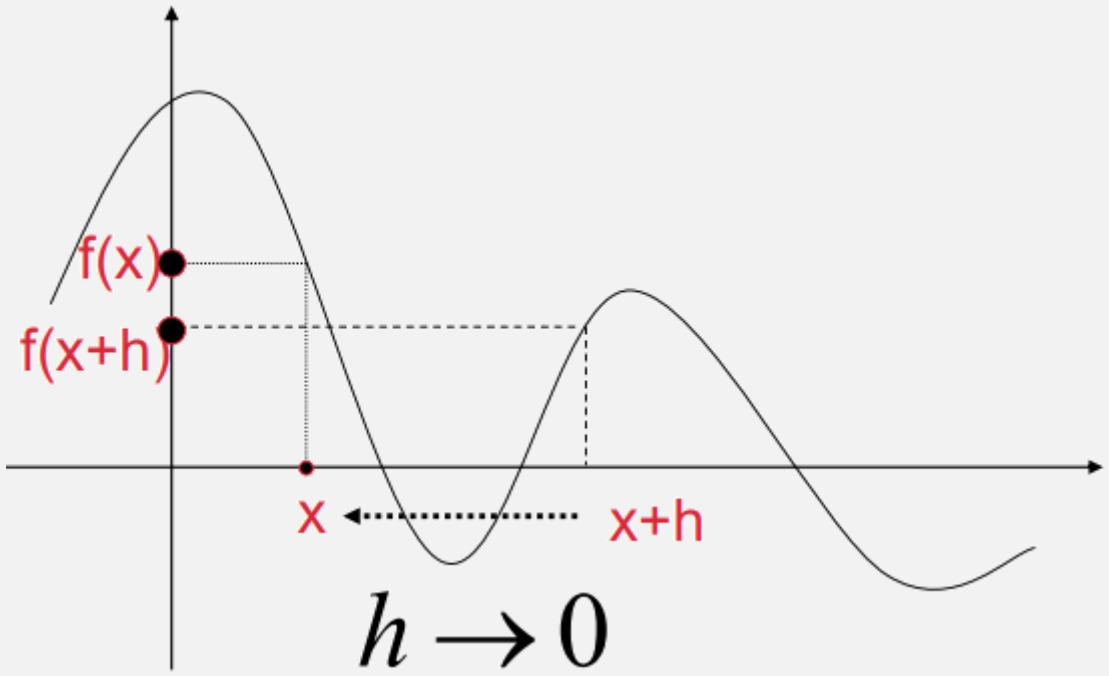
$$\Delta x = x_0 + h - x_0$$
$$\Delta f = f(x_0 + h) - f(x_0)$$

the ratio between the increment of the variable x passing from x_0 to $x_0 + \mathrm{h}$ and the increment of the function f is called difference quotient of the function relating to the passage from x_0 to $x_0 + \mathrm{h}$

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

the difference quotient expresses the "variability of f" respect to a certain interval

Difference quotient → pointwise information



Pointwise information about monotonicity at x_0 requires considering a very small interval $[x_0, x_0 + h]$, which is obtained when the point $x_0 + h$ gets «closer» to x_0, that is to say as $h \to 0$.

Definition of the Derivative

Let $f(x)$ be defined on $[a, b]$ and let x_0 be an interior point.

The derivative of $f(x)$ at x_0 is defined as the limit, if it exists and is finite, of the difference quotient of $f(x)$ passing from x to $x + h$, that is to say as h $h \to 0$:

$$D\big(f(x_0)\big) = \frac{df(x_0)}{dx} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



Derivative of the function f in x

Definition of the Derivative

$$D\big(f(x_0)\big) = \frac{df(x_0)}{dx} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The derivative of a function $f(x)$ at an interior point x_0 is a real number, when it exists.

If the function $f(x)$ is defined in an interval $[a, b]$, at the endpoints of the interval only the right derivative (at a) and the left derivative (at b) can be defined.

A function is differentiable on an interval $[a, b]$ if it is differentiable at every interior point of $[a, b]$ and if admits a right derivative at a and a left derivative at b.

Differentiability and Continuity

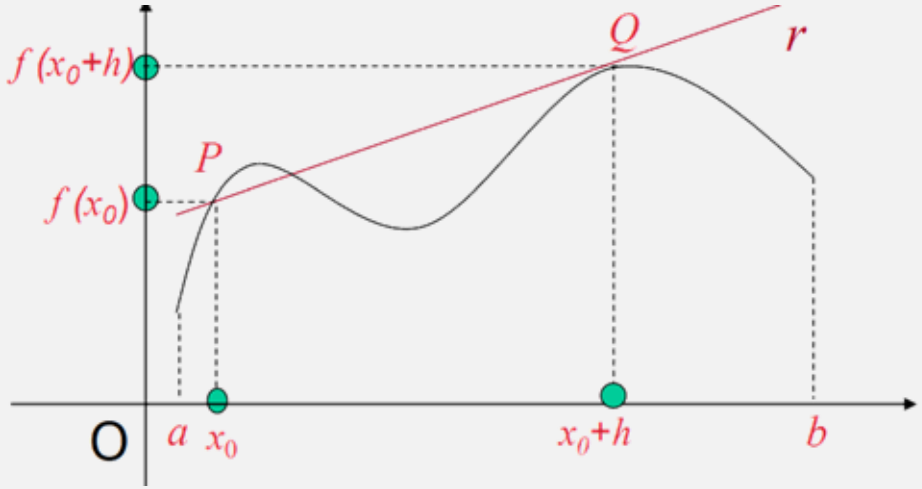
If a function $f(x)$ is differentiable in a point $x_0 \in (a, b)$, then $f(x)$ is continuous in x_0

from this theorem, it immediately follows that: in the points of discontinuity, a function does not admit derivative (it is not differentiable)

therefore, if $f(x)$ is not continuous in $x_0 \in (a, b)$, then $f(x)$ is not differentiable in x_0

Geometric Meaning of the Derivative

Let $f(x)$ be a function differentiable on an interval $[a,b]$ and let x_0 be a fixed interior point of the interval $[a,b]$.
We move from the point x_0 to another point x_0+h inside the interval $[a,b]$ so that the corresponding values of the function f can be considered.



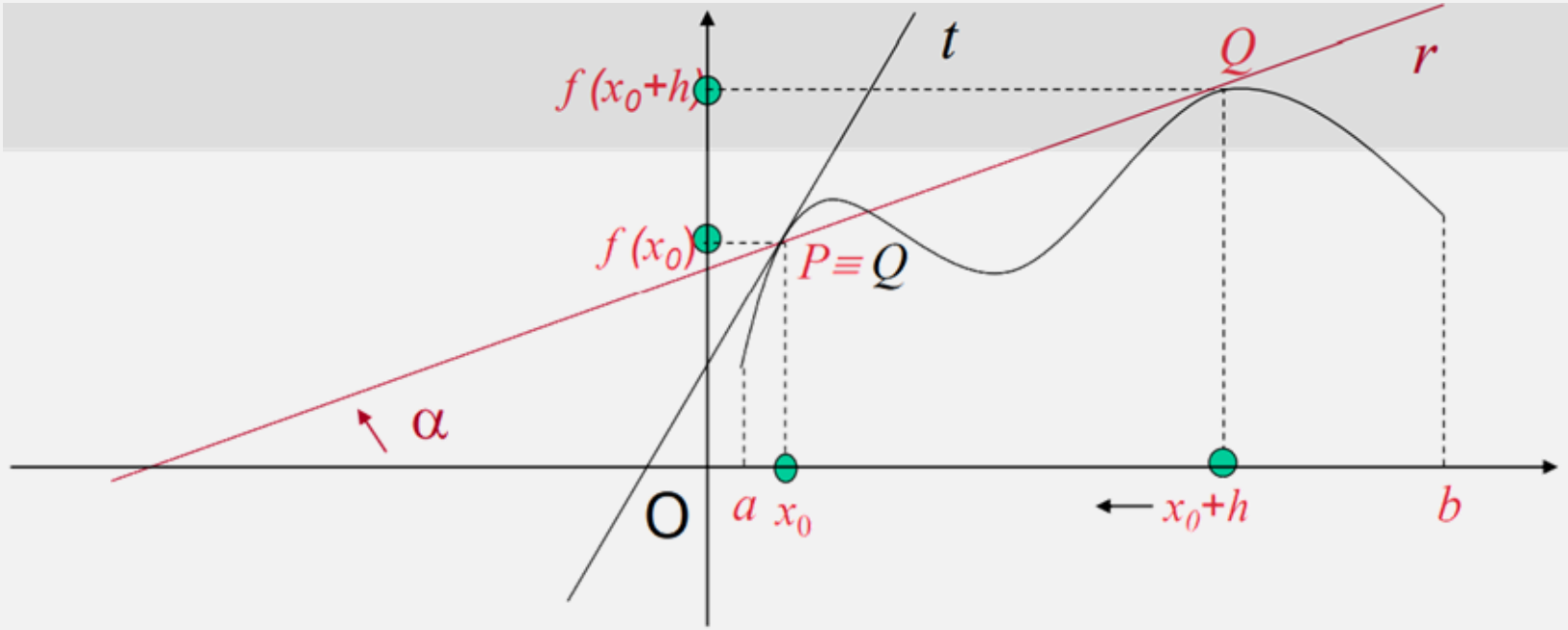
we indicate the two points belonging to the graph of f as:

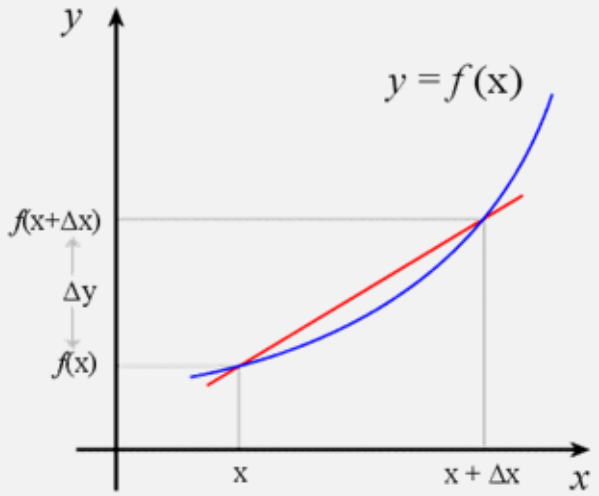
$$P\left(x_0, f\left(x_0\right)\right)$$
$$Q\left(x_0+h, f\left(x_0+h\right)\right)$$

Let r be the line passing through P and Q and forming an angle α with the positive semi-axis of x values

Geometric Meaning of the Derivative

These variations, as $h \to 0$, cease when point Q coincides with point P.
At that moment, the line passing through P and Q approaches a limiting position, which is identified as the tangent line to the graph of the function f at the point with abscissa x_0 that is, at point P.





Geometric Meaning of the Derivative

Therefore, if α is the angle formed by the tangent line to the graph of f at the point with abscissa x_0, and m_t is its slope, then:

$$f'(x_0) = \tan\alpha = m_t$$

That is:

The derivative $f'(x_0)$ of the function f in the point x_0 is equal to the angular coefficient m_t of the tangent line to the graph at the point $P\big(x_0, f(x_0)\big)$

Derivatives of Elementary Functions

Derivative of the Power Function

Let the function $f(x) = x^{\alpha}, \alpha \in \mathbb{R}, x > 0$.

It can be shown that the derivative of this function, for all $x > 0$ ($\forall x \in \mathbb{R}^{+}$) and for all $\alpha \in \mathbb{R}$, is:

$$Dx^{\alpha} = \alpha x^{\alpha-1}, \forall x > 0, \forall \alpha \in \mathbb{R}$$

Example.
$f(x) = x^2$ in $x_0 = 2$

$$\frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{(2+h)^2 - (2)^2}{h} = \frac{4 + 4h + h^2 - 4}{h} = \frac{h^2 + 4h}{h} = h + 4$$

$$\Rightarrow \lim_{h \to 0} h + 4 = 4$$

The function $f(x) = x^2$ is differentiable in $x_0 = 2$ and $f'(2) = 4$

Derivatives of Elementary Functions: Differentiation Rules

$$Da^x = a^x \log a$$

$$De^x = e^x$$

$$D \log_a x = \frac{1}{x} \cdot \frac{1}{\log a}$$

$$D \log x = \frac{1}{x}$$

$$D \sin x = \cos x$$

$$D \cos x = -\sin x$$

Derivatives of Elementary Functions

$f(x)$	$f'(x)$
k	0
x	1
x^n	nx^{n-1}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$	$a_1 + 2a_2x + \cdots + na_nx^{n-1}$
$\log_a x$	$\frac{1}{x} \cdot \log_a e = \frac{1}{x} \cdot \frac{1}{\ln a}$
$\log_e x$	$\frac{1}{x} \cdot \log_e e = \frac{1}{x}$
a^x	$a^x \log_e a = a^x \cdot \frac{1}{\log_a e}$
e^x	$e^x \log_e e = e^x$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\lvert x \rvert$	$\frac{x}{\lvert x \rvert} = \frac{\lvert x \rvert}{x}$
$\lvert f(x) \rvert$	$f'(x)$ with $f(x) > 0$ $-f'(x)$ with $f(x) < 0$

Second Derivative – n-th Derivative

At this point, it is natural to ask whether the derivative function $f'(x)$ is itself differentiable at a point or on the entire interval (a, b).
If this is the case, we call the derivative of f' the second derivative, denoted by:

$$f''(x)$$

$$D^2 f(x)$$

Similarly, we define the third derivative $f'''(x)$, fourth derivative $f^{IV}(x)$, and in general the nth derivative:

$$f^n(x)$$

$$D^n f(x)$$

Differentiation Rules

Differentiation Rules (Sum, Product, Quotient)

Let $f : (a, b) \to \mathbb{R}, g : (a, b) \to \mathbb{R}$ be two functions differentiable on (a, b). Then the functions:

$$f \pm g, \quad f \cdot g, \quad \frac{f}{g} \text{ (with } g \neq 0)$$

are differentiable in (a, b), and the following formulas hold:

- $(f \pm g)' = f' \pm g'$
- $(f \cdot g)' = f' \cdot g + f \cdot g' \to (k \cdot f)' = k \cdot f'$
- $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \to \left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

Exercises. Compute the derivative of the following functions

$f(x) = 4x^2 + 4$

$\frac{d}{dx}(4x^2) + \frac{d}{dx}(4) =$

$f'(x) = 4 \cdot 2x^{2-1} + 0 = 8x$

$f(x) = \sqrt{x} + x^{2/3}$

$\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}\left(x^{\frac{2}{3}}\right)$

$f'(x) = \frac{1}{2}x^{-1/2} + \frac{2}{3}x^{-1/3} = \frac{1}{2x^{1/2}} + \frac{2}{3x^{1/3}}$

$f(x) = -x^{-2} + 3x^3$

$\frac{d}{dx}(-x^{-2}) + \frac{d}{dx}(3x^3) =$

$f'(x) = -\ -2x^{-2-1} + 3 \cdot 3x^{3-2} = 2x^{-3} + 9x^2$

Theorem: Derivative of Composite Functions (Chain Rule)

Let the composite function $f\big(g(x)\big)$ be defined by two functions $f(x)$ and $g(x)$.

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f\big(g(x)\big)$ is differentiable and the followinf formula holds:

$$(f \circ g)' = f'\big(g(x)\big) \cdot g'(x)$$

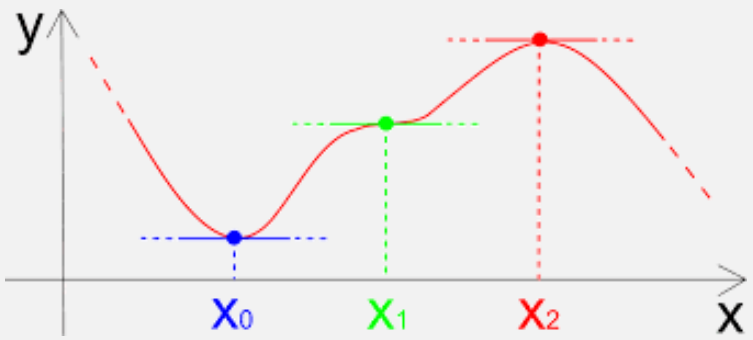
Stationary Point

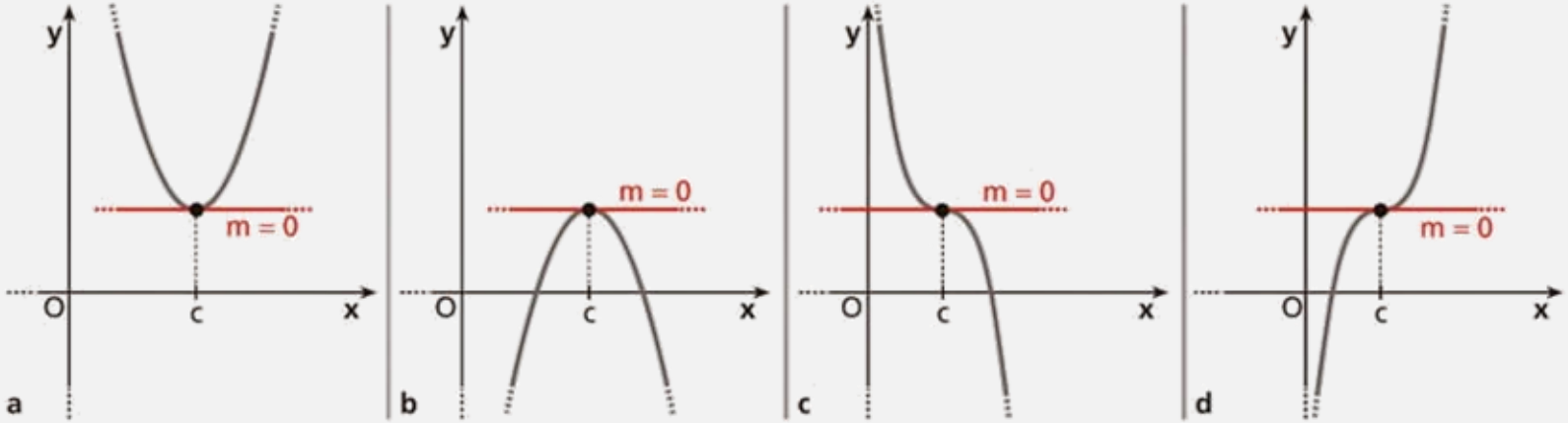
Points x at which the derivative of a function f vanishes are called stationary points of f.

That is, x_0 such that:

$$f'(x_0) = 0$$

Equivalently, at such a point the tangent line is horizontal.





Characterization of Monotone Functions

Monotonicity Criterion

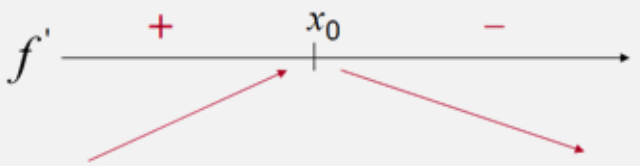
Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b).
Then:

- f increasing on $[a, b] \Leftrightarrow f'(x) \geq 0, \forall x \in (a, b)$

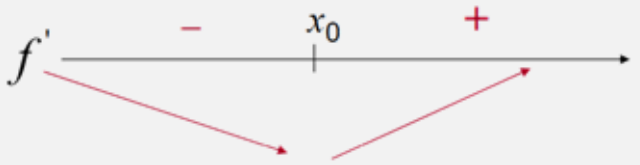
- f decreasing on $[a, b] \Leftrightarrow f'(x) \leq 0, \forall x \in (a, b)$

From the monotonicity criterion, it is possible to study relative and absolute maxima and minima of a function.

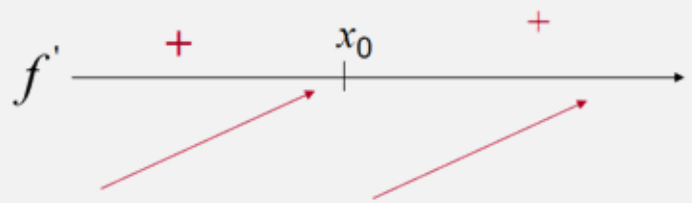
More precisely, if $f'(x_0) = 0$, one can verify that:



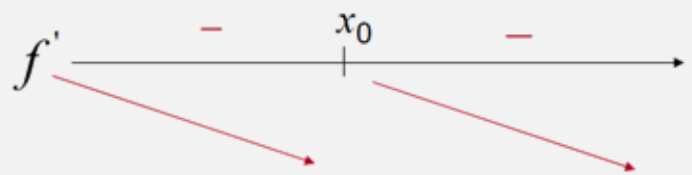
x_0 point of local minimum



x_0 point of inflection



x_0 point of inflection



x_0 point of local maximum

Having found possible local extrema, compute f at those points and compare with $f(a)$ and $f(b)$.

Example – The Peak: Where the Derivative Is Zero

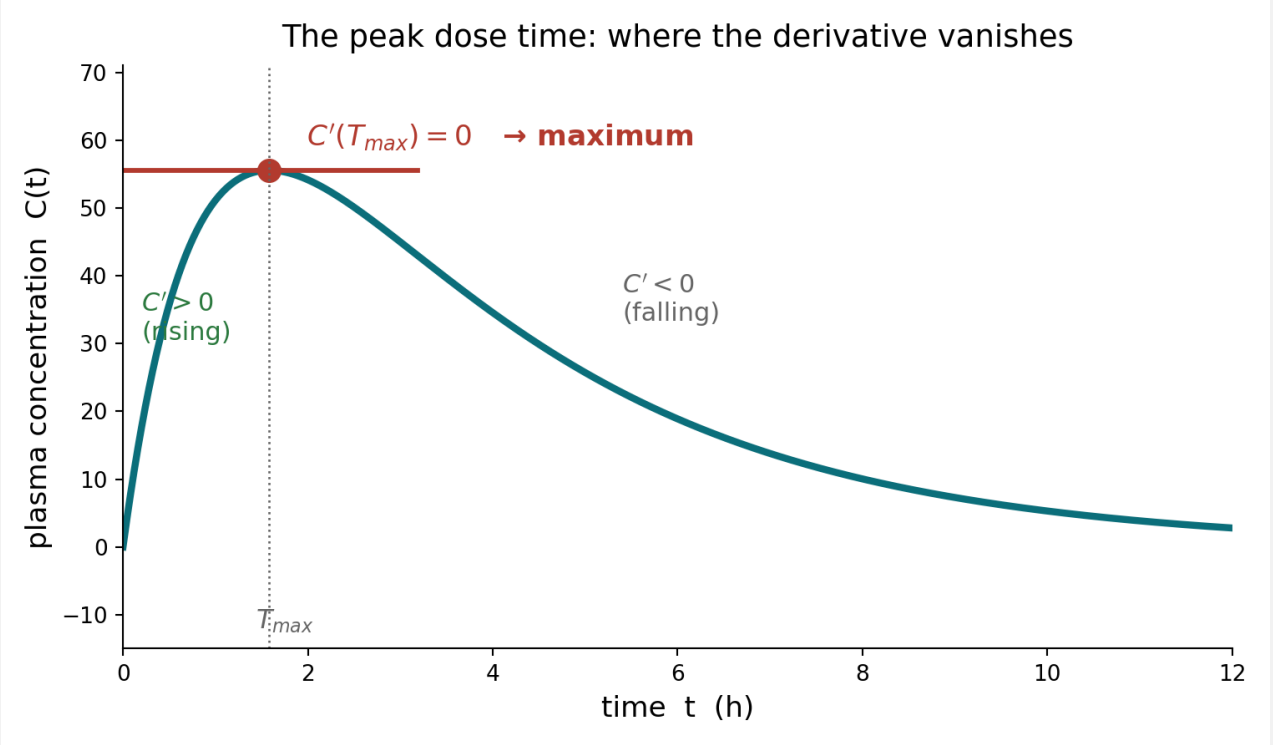
After a dose, plasma concentration rises, reaches a peak, then falls.

At the peak time T_{max} the instantaneous rate of change is zero:

$$C'(T_{max}) = 0$$

Before T_{max} the derivative is positive (rising); after it, negative (falling).

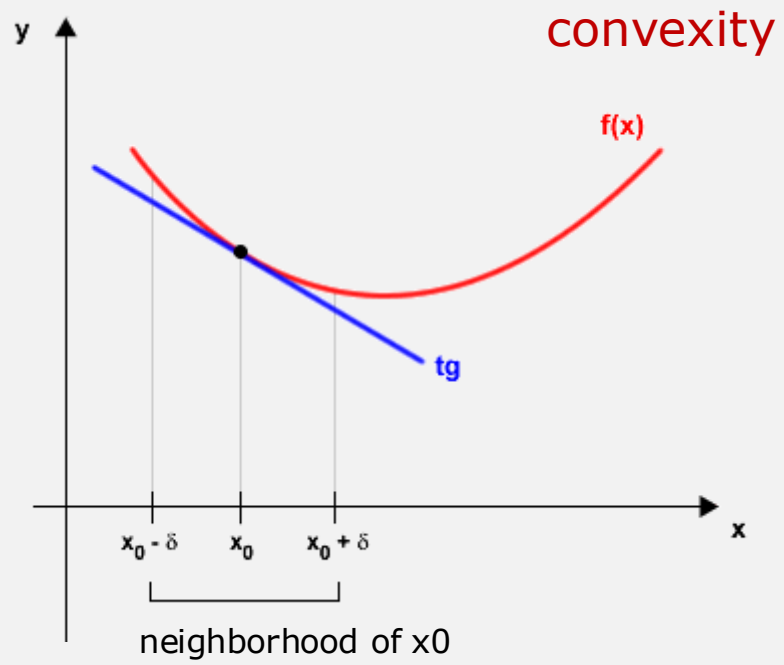
Setting the derivative to zero locates the maxima and minima of any biomedical curve.



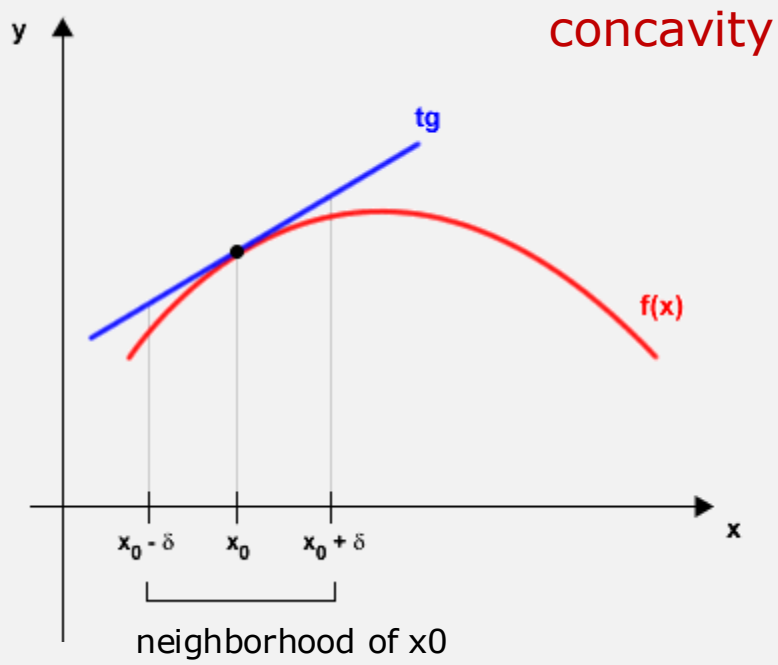
Concavity and Convexity

Convex and Concave Functions

A function $f(x)$ is said to be convex on $[a,b]$ if $\forall x_0 \in [a,b]$, the graph of $f(x)$ lies above the tangent line to the graph at the point $(x_0, f(x_0))$.



A function $f(x)$ is said to be concave on $[a,b]$ if $\forall x_0 \in [a,b]$, the graph of $f(x)$ lies below the tangent line to the graph at the point $(x_0, f(x_0))$.



Convexity Criterion

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b).

Then:

- f convex on $[a, b] \Leftrightarrow f'$ increasing on $[a, b] \Leftrightarrow f'' \geq 0 \, on \, [a, b]$

- f concave on $[a, b] \Leftrightarrow f'$ decreasing on $[a, b] \Leftrightarrow f'' \leq 0 \, on \, [a, b]$

If the first derivative of a function is increasing on $[a, b]$, the function $f(x)$ is convex
If the first derivative of a function is decreasing on $[a, b]$, the function $f(x)$ is concave

If the first derivative represents the slope of the graph of the function, the second derivative represents the variation of that slope.

Example – The Inflection Point Is the Fastest Growth

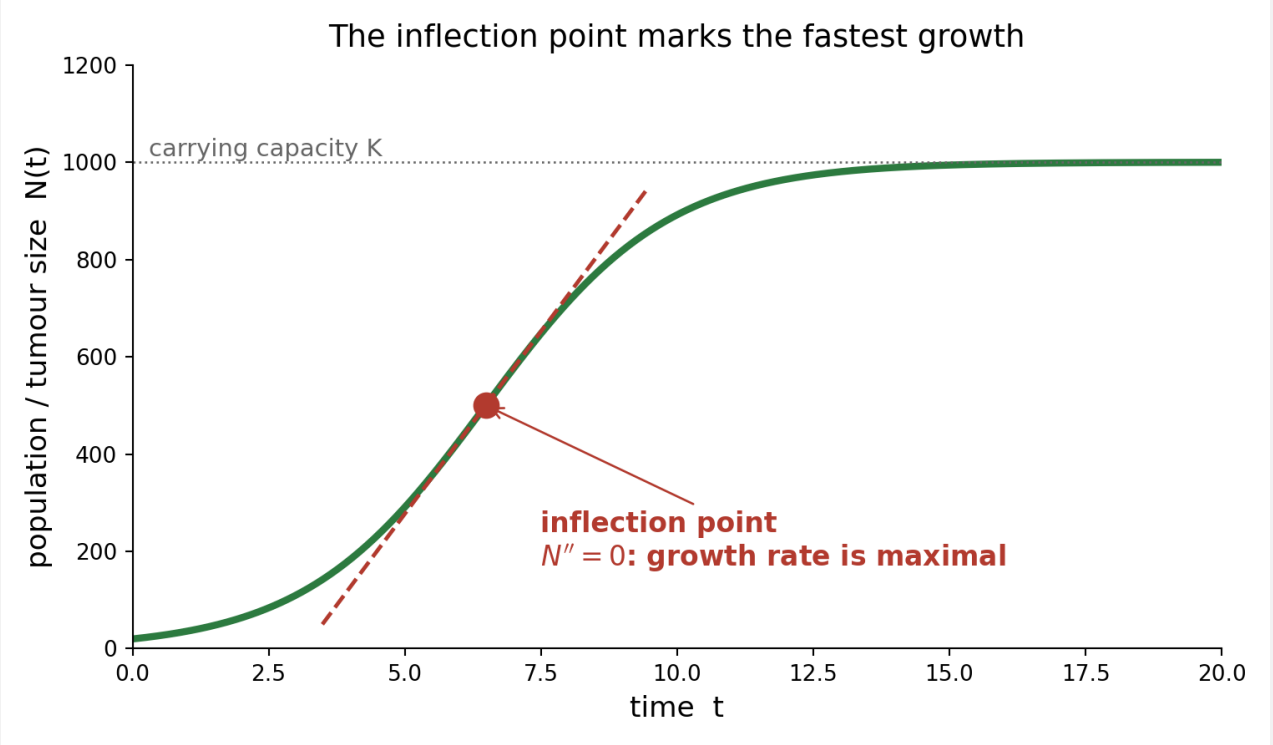
A tumour or population grows slowly, then rapidly, then slows near its limit.

The steepest point of the curve is the inflection point, where the second derivative is zero:

N''(t) = 0 → the growth rate N'(t) is maximal

For logistic growth it occurs at about half the carrying capacity.

The second derivative reveals when a process accelerates and when it saturates.



The Study of a Function

$$f(x) = \frac{x^3}{x^2 - 4}$$

1. DOMAIN OF f

The first important piece of information is to determine for which real values the function is defined.

In this example, the domain is given by $D_f = \mathbb{R} - \{\pm 2\}$, that is, it consists of all real numbers except those that make the denominator equal to zero (± 2).

Recall:

- For rational functions, the domain is given by those points for which the denominator is nonzero;
- For algebraic functions involving even roots, the domain consists of those values for which the radicand is non-negative;
- For functions involving a logarithm, the argument must be positive;
- For exponential, trigonometric, and polynomial functions, the domain coincides with the whole real line.

$$f(x) = \frac{x^3}{x^2 - 4}$$

2. INTERSECTIONS WITH THE AXES

Determining the points of intersection between the curve and the coordinate axes means finding:

- the coordinates of the points where the function intersects the x-axis, and
- the coordinates of the points where the function intersects the y-axis.

- To find the points of intersection with the y-axis, we evaluate the function at $x = 0$.
- To find the points of intersection with the x-axis, we determine the values of x for which $y = 0$.

In this example, the function intersects both the x-axis and the y-axis only at the origin.

$$f(x) = \frac{x^3}{x^2 - 4}$$

3. SIGN OF f

We want to determine the values of x for which $f(x) \geq 0$. In the example, we need to study separately the sign of the numerator and of the denominator, and then combine the obtained information. The numerator is positive or null for $x \geq 0$, while the denominator is positive for $x < -2$ and $x > 2$. Hence, we obtain that $f(x) \geq 0$ for $-2 < x \leq 0, x > 2$.

Recall:

- Studying the sign of a rational function amounts to solving two inequalities, one for the numerator and one for the denominator, and then combining the results;
- An exponential function is always strictly positive;
- To study the sign of a logarithmic function, it is necessary to determine for which values of x the argument of the logarithm is greater than or equal to 1;
- For odd roots, the sign is determined by the sign of the radicand;
- Even roots are always non-negative;
- For trigonometric functions, it is useful to refer to the unit circle to determine their sign.

$$f(x) = \frac{x^3}{x^2 - 4}$$

4. ASYMPTOTES OF f

Recall:

- if $\lim_{x \to x_0^+} f(x) = \pm\infty$, then, the line $x = x_0$ is called <u>vertical asymptote</u> for $f(x)$
- if $\lim_{x \to +\infty} f(x) = l_1$, $\lim_{x \to -\infty} f(x) = l_2$, then $f(x)$ admits two <u>horizontal asymptotes</u>: $y = l_1$ to $+\infty$ and $y = l_2$ to $-\infty$

- If the following conditions hold, then the graph of the function $f(x)$ admits oblique asymptote of equation $y = mx + q$: $\lim_{x \to \pm\infty} f(x) = \pm\infty$, $\lim_{x \to \pm\infty} \frac{f(x)}{x} = m \neq 0 \; finite$, $\lim_{x \to \pm\infty} [f(x) - mx] = q \; finite$

$$f(x) = \frac{x^3}{x^2 - 4}$$

5. MAXIMA AND MINIMA

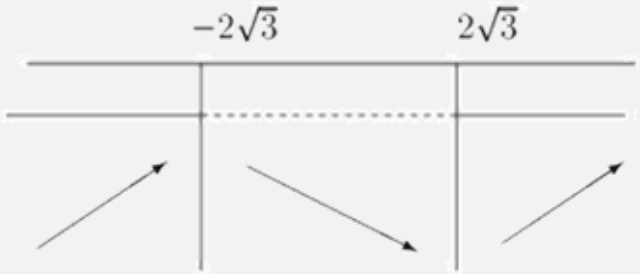
To compute maxima and minima of a function, it is necessary to compute the first derivative and study its sign.

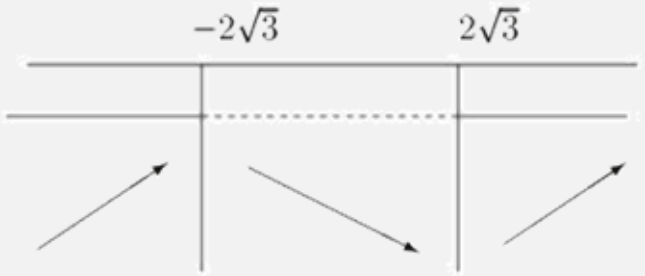
$$f'(x) = \frac{x^2(x^2 - 12)}{(x^2 - 4)^2} \geq 0 \Longrightarrow x \leq -2\sqrt{3}, x \geq 2\sqrt{3}$$

Hence, in intervals where $f'(x) > 0$ the function is increasing, while in intervals where $f'(x) < 0$ the function is decreasing.

The derivative vanishes at$x = \pm 2\sqrt{3}$. To determine whether these points are maxima, minima, or points of inflection, we study the sign of the derivative. Thus:

- the function is increasing on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$
- the function is decreasing on $(-2\sqrt{3}, 2\sqrt{3})$





The points at which the derivative vanishes $x = \pm 2\sqrt{3}$. The point $x = -2\sqrt{3}$ is a local minimum, while $x = 2\sqrt{3}$ is a local maximum.

We also observe the presence of a point of inflection at $x = 0$, that is, a point at which the first derivative vanishes but which is neither a maximum nor a minimum.
Therefore, at $x = 0$, the graph of f has a horizontal tangent line.

$$f(x) = \frac{x^3}{x^2 - 4}$$

6. CONCAVITY

Studying the sign of the second derivative $f''(x)$, that is, the derivative of the first derivative, allows us to analyze the concavity of the function f. in fact, if $f''(x) > 0$, the function is convex; if $f''(x) < 0$, the function is concave.

In this example: $f''(x) = \frac{8x(x^2+12)}{(x^2-4)^3} \geq 0 \rightarrow -2 < x \leq 0, x > 2$

If $f''(x) > 0$, the function $f'(x)$ is increasing, hence the tangent line to the graph has an increasing slope; if $f''(x) < 0$, the function $f'(x)$ is decreasing, hence the tangent line to the graph has a decreasing slope;

$$f(x) = \frac{x^3}{x^2 - 4}$$

7. GRAPH

At this point, it is necessary to combine all the information obtained in the previous steps and draw the graph of the function f.

