

Chapter 2 — Functions

Definitions • examples • graphs • applications

Chapter roadmap

1. General concept of function in set-theoretic language.
2. Sequences: functions defined on the natural numbers.
3. Linear and quadratic functions; direct, quadratic, and inverse proportionality.
4. Composition and inverse functions.
5. General graph properties: boundedness, monotonicity, convexity.
6. Elementary functions: powers, exponentials, logarithms, trigonometric functions.

Dependence as the starting point

Examples:

“Temperature depends on altitude.”

“the price of rice depends on the quantity supplied.”

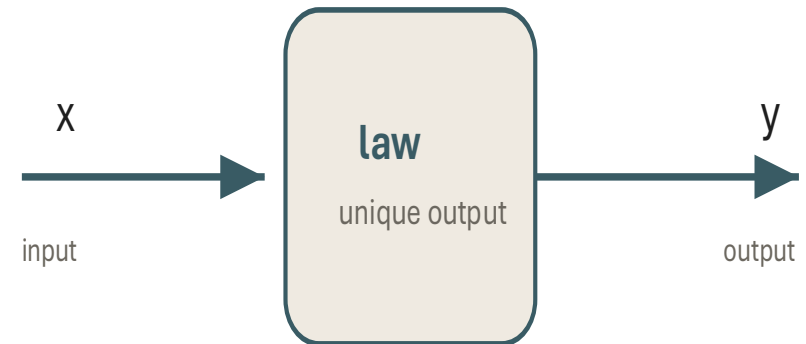
Definition of function: Given two sets **A** and **B**, a **function from A to B** is a rule that assigns **exactly one element of B** to each element of A.

A is called the **domain**, while **B** is called the **codomain**.

An essential feature of a function is that each input must have a **unique output**.

For this reason, a function is said to define a **unique correspondence from A to B**.

Function as a black box



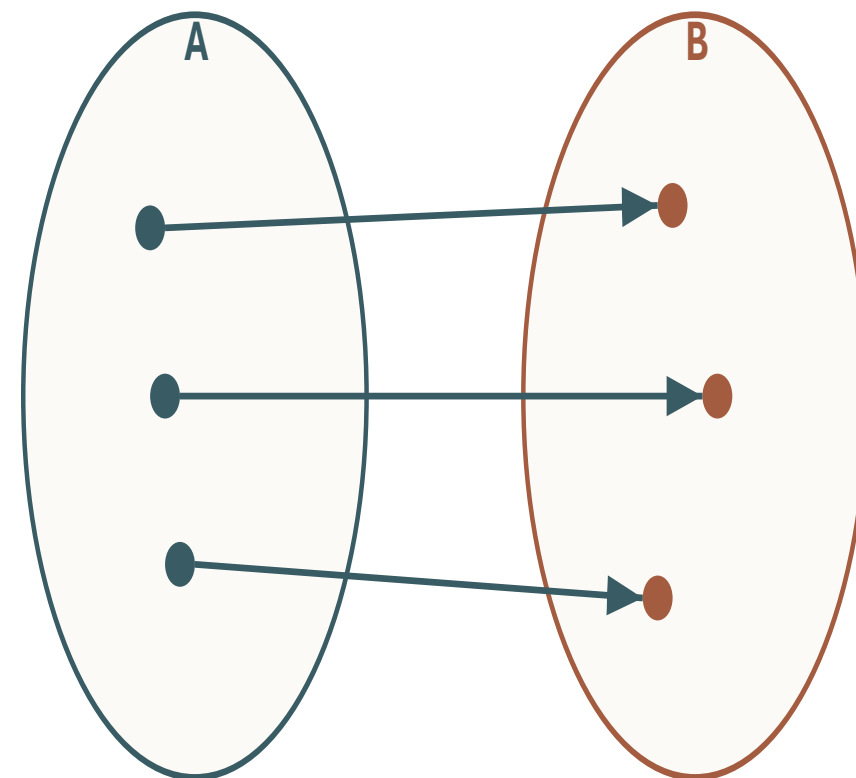
Example: Polygons-Area

Let A be the set of polygons in a plane.

Let B be the set of real numbers.

The correspondence “polygon $\mapsto$ area” is a function from A to B .

- A function is usually denoted by letters such as f , g , h , F , G .
- The notation $f: A \rightarrow B$ is read “ f from A to B .”
- If $x \in A$ and $y \in B$, the notation $f: x \mapsto y$ is read “ *f maps x to y .*”
- The element y is also denoted $f(x)$, read “ f of x .”
- y is the *image* of x through f .
- x is a *preimage* of y .
- Collecting all outputs obtained from inputs in A gives the image of A , denoted $f(A)$. $f(A) = \{ f(x) : x \in A \}$



every element of A gets exactly one image in B

Independent and dependent variables - Definition of graph

The input x is called the *independent variable*, while the output y is *the dependent variable*.

We focus on **real-valued functions of a real variable**:

$$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$$

Examples: production cost vs. quantity, oil production vs. time, car speed vs. position.

To understand how a function behaves, it is useful to represent it graphically on a **Cartesian plane**.

Definition of graph: The graph of a function $f: A \rightarrow B$ is the set of all ordered pairs (x, y) such that:

$$x \in A, y \in B, \text{ and } y = f(x). \\ \text{Graph}(f) = \{ (x, y) : x \in A, y \in B, y = f(x) \}$$

A circle cannot be the graph of a function

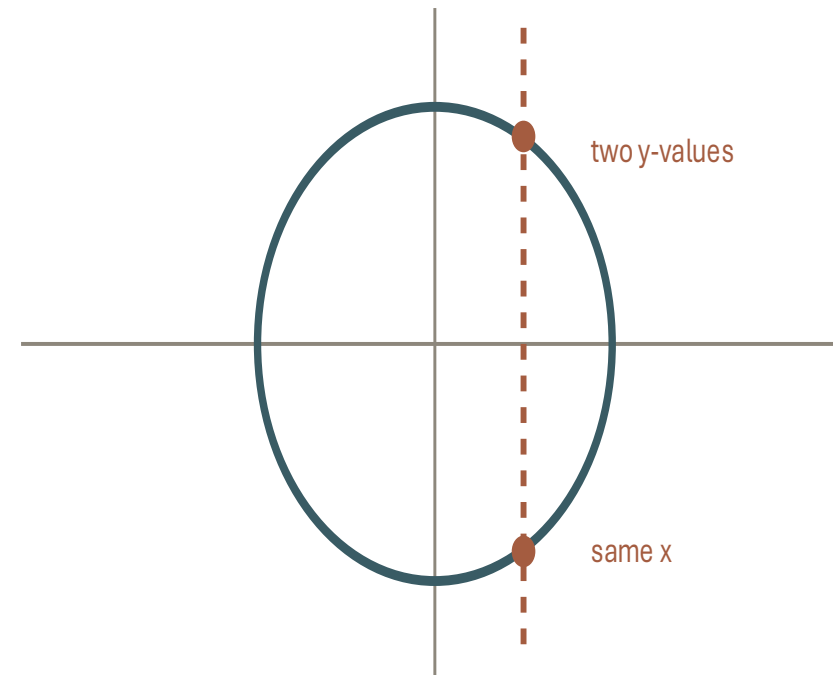
- Two functions are equal if they have the same graph.
- This implies that they have the same domain and the same image.
- Geometric criterion: every vertical line intersects the graph of a function at most once.

When a function is given by an equation $y = f(x)$, (unless otherwise stated), its domain is the largest subset of $\mathbb{R}$ where the operations in $f(x)$ are possible.

This is called the natural domain.

If the codomain is not specified, it is understood to be $\mathbb{R}$.
natural domain = largest subset of $\mathbb{R}$ where $f(x)$ is meaningful

Not every curve is a function



Examples

1. $f(x) = 2x^3 - 5x^2 + 7$

Domain: $\mathbb{R}$

Polynomial functions are defined for every real number.

2. $f(x) = 1 / (x - 3)$

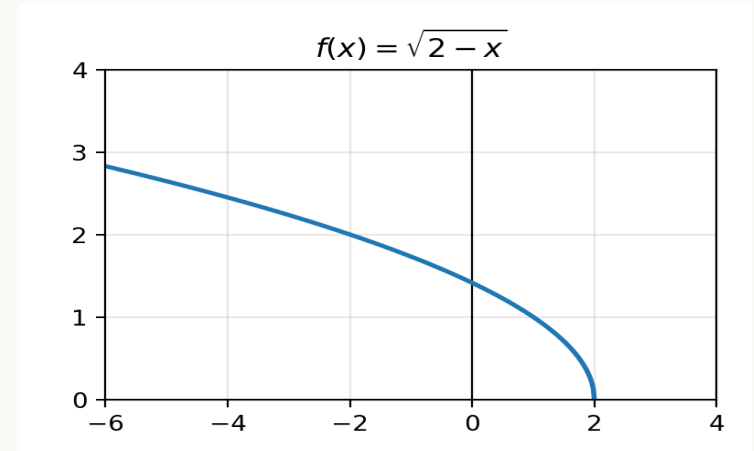
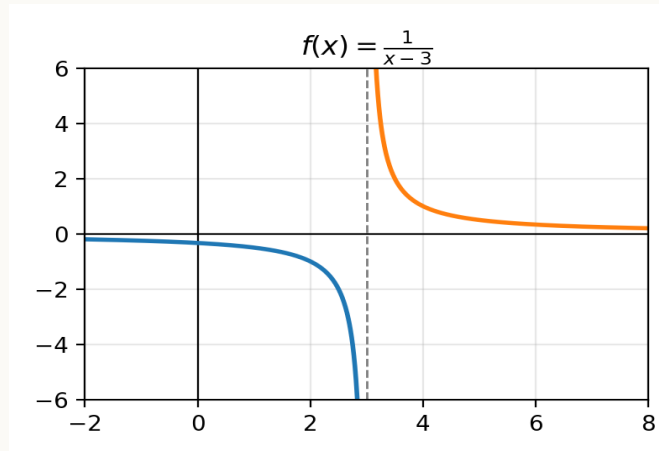
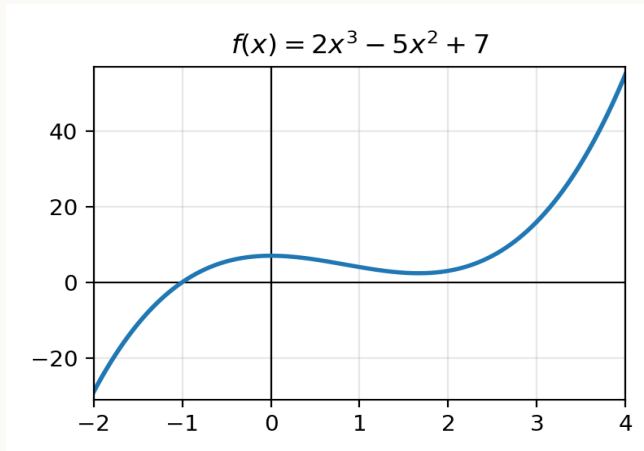
Domain: $\mathbb{R} \setminus \{3\}$

The function is not defined at $x = 3$ because the denominator cannot be zero.

3. $f(x) = \sqrt{2 - x}$

Domain: $(-\infty, 2]$

The square root requires $2 - x \geq 0$, so $x \leq 2$.



Continuous and Discrete Variables

Some quantities vary **continuously** and are naturally represented by real numbers. Examples include the amount of **liquid in a tank**, the **distance travelled by a moving object**, or **time**.

Other quantities are **discrete** and are better represented by natural numbers. For example, a company may count the **number of boxes of biscuits or bottles of wine produced**. In these cases, the mathematical model is a **sequence**, that is, a function defined on $\mathbb{N}$.

Time can also be treated as a discrete variable when observations are made at regular intervals. In this case, each observation corresponds to a natural number.

Real-valued sequence

A **sequence** is an ordered list of elements, where each element is associated with a natural number. The element in position n is denoted by $\mathbf{a_n}$:

$$\mathbf{n} \rightarrow \mathbf{a_n}$$

Here, $\mathbf{a_n}$ is the general term of the sequence, while $\mathbf{n}$ is the index. The sequence is denoted by $\{\mathbf{a_n}\}$.

Definition

A **real-valued sequence** is a function that assigns a real number to each natural number:

$$\mathbf{a: \mathbb{N} \rightarrow \mathbb{R}}$$

The value $\mathbf{a_n}$ is called the **n-th term** of the sequence.

Sequences are often defined by an **explicit formula**, which gives a_n directly in terms of n .

The index does not necessarily have to start from 0 or 1: a sequence may be defined from any natural number $\mathbf{n_0}$ onward.

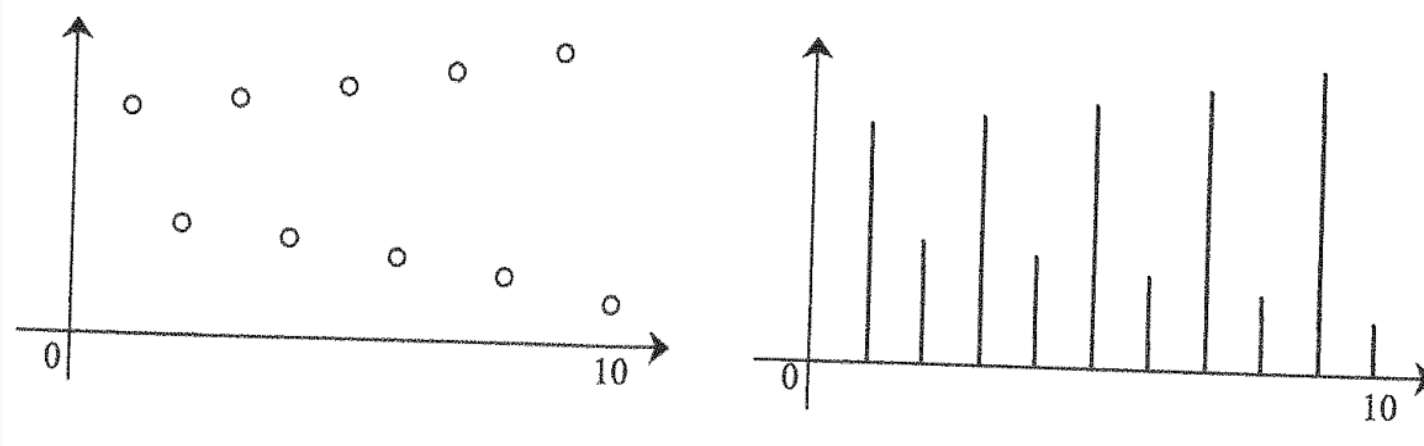
Graphical Representation of a Sequence

A sequence $\{a_n\}$ can be represented on a Cartesian plane by placing the index values $n = 0, 1, 2, 3, \dots$ on the horizontal axis and the corresponding sequence values $a_0, a_1, a_2, a_3, \dots$ on the vertical axis.

The graph of the sequence is therefore formed by the discrete points:

$$(n, a_n)$$

Since the index takes only natural values, the graph is **discrete**, not continuous.



Recursive sequences

A sequence can be defined not only by an explicit formula, but also **recursively**.

A recursive definition usually requires two elements:

- **initial value** a_0 ;
- **recurrence relation** that determines the next term a_{n+1} from the previous term a_n .

In this way, each new term is generated step by step from the one before it.

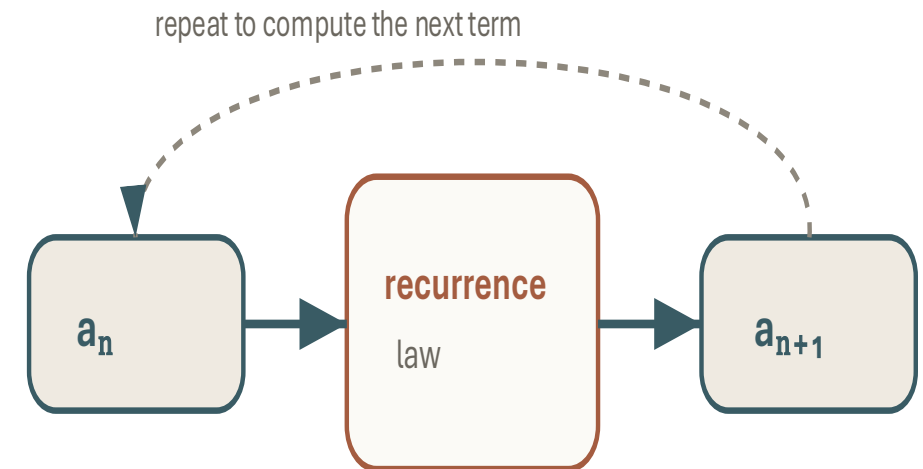
Example – Arithmetic Sequence

$$a_0 = a$$

$$a_{n+1} = a_n + r$$

where r is the common difference.

$$a_n = a + nr$$



Example — Simple Interest Capitalization

A capital **C** is invested for **n years** at a simple annual interest rate **$r > 0$** .

Each year, the interest earned is constant:

$$\mathbf{Cr}$$

Therefore, after **n years**, the accumulated amount is:

$$\mathbf{a_n = C(1 + rn)}$$

This gives a sequence where each term represents the value of the capital after n years.

Recursive Definition

The same sequence can also be defined recursively:

$$\mathbf{a_0 = C}$$

$$\mathbf{a_{n+1} = a_n + Cr}$$

So, each year the capital increases by the same fixed amount **Cr**.

Geometric sequence

In many economic applications, we study **sequences $\{a_n\}$ in which the ratio between consecutive terms is constant**. These are called *geometric (or exponential) sequences*, and the constant ratio is called the common ratio.

The general term of a geometric sequence is:

$$a_n = aq^n$$

where **$a = a_0 \neq 0$ is the initial term and q is the common ratio**.

A key property of a geometric sequence is that the **percentage change between consecutive terms is constant**.

In fact,

$$(a_{n+1} - a_n) / a_n = q - 1$$

If we let

$$p = q - 1$$

then **p** is the *constant growth rate*, and the general term can be written as:

$$a_n = a(1 + p)^n$$

Applications

- population growth or decline over time;
- company sales growing at a constant percentage rate;

Compound Interest

Compound capitalization is a typical example of a geometric sequence.

If a capital C is invested at an annual interest rate r , and the interest is reinvested every year, then:

$$C_1 = C(1 + r)$$

After two years:

$$C_2 = C(1 + r)^2$$

and, in general, after n years:

$$C_n = C(1 + r)^n$$

So compound interest generates a geometric sequence with common ratio $1 + r$.

Direct proportionality and Linear Functions

The **simplest** relationship between two quantities is **direct proportionality**.

If x is the *quantity purchased*, m is the *unit price*, and y is the *total cost*, then:

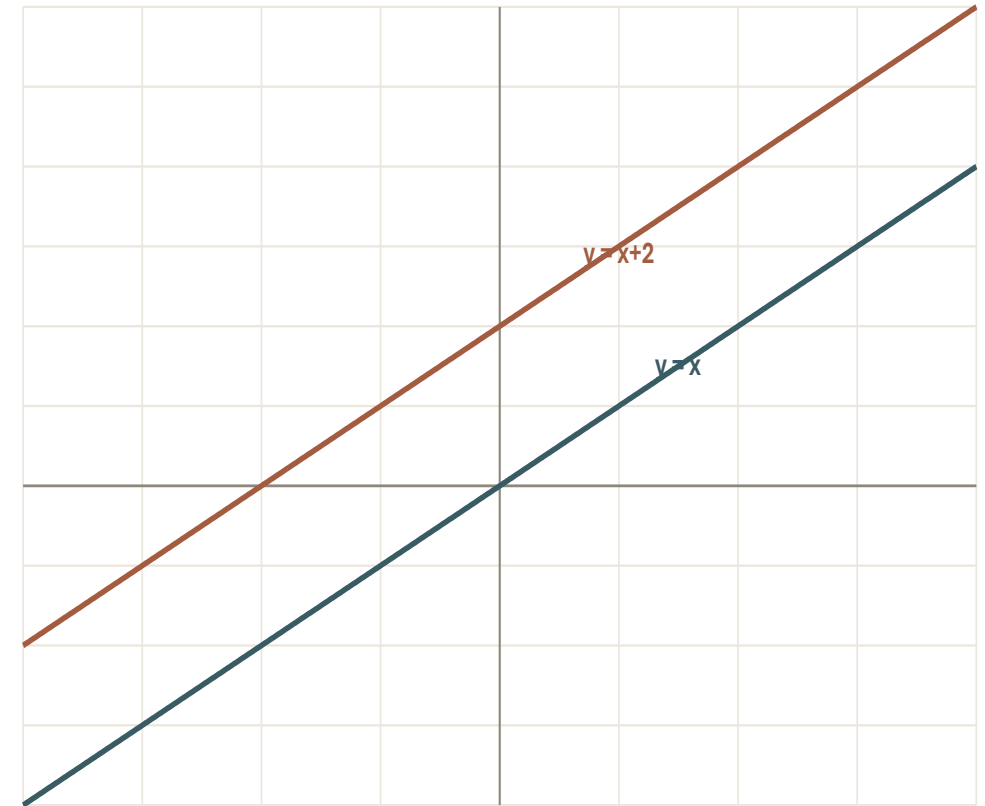
$$y = mx$$

The ratio between the two variables is constant:

$$y / x = m$$

where m is the **constant of proportionality**.

Linear and affine linear graphs



Linear function

Definition: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called linear if, for every $x_1, x_2, \alpha \in \mathbb{R}$:

- $f(x_1 + x_2) = f(x_1) + f(x_2)$
- $f(\alpha x) = \alpha f(x)$

Every linear function from $\mathbb{R}$ to $\mathbb{R}$ has the form: $f(x) = mx$. Its graph is a straight line passing through the origin.

Affine Functions

A more general straight-line function has the form:

$$f(x) = mx + q$$

These functions are called affine functions.

m = slope

$q = f(0)$ = y-intercept

The graph is obtained by shifting the line $y = mx$ vertically by q units.

Economic example- Market equilibrium

The quantity demanded and the quantity supplied of a good depend on its price.

Let:

$q_d(p)$ = quantity demanded

$q_s(p)$ = quantity supplied

p = price, with $a, b, c, d > 0$.

Demand

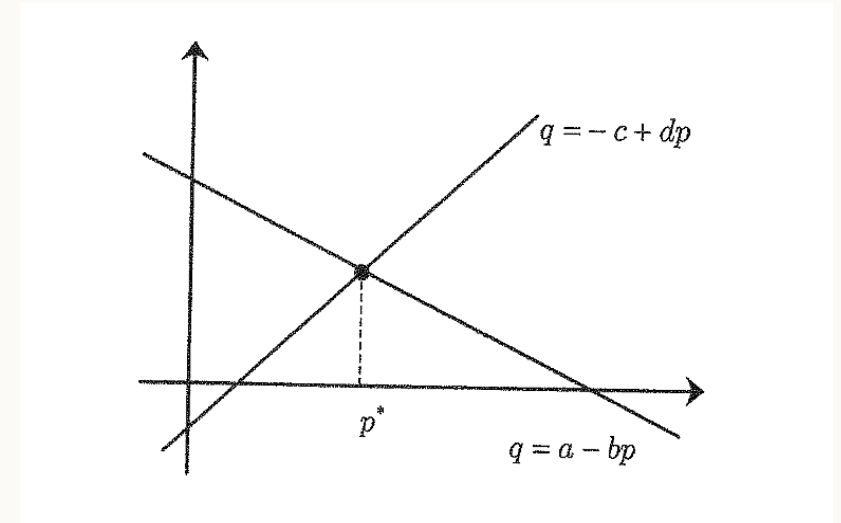
$$q_d(p) = a - bp$$

As the price increases, the quantity demanded decreases.

Supply

$$q_s(p) = -c + dp$$

As the price increases, the quantity supplied increases.



The market is in equilibrium when quantity demanded equals quantity supplied:

$$q_d(p) = q_s(p)$$

Therefore:

$$a - bp = -c + dp$$

Solving for p gives the **equilibrium price**:

$$p^* = (a + c) / (b + d).$$

Quadratic proportionality

A quantity is **quadratically proportional** to another when it depends on its square.

Example: Consider a rectangular store such that the ratio of the side lengths is $a = 0.618$. If x is the length of the longer side, the store area is:

$$A(x) = ax^2$$

Therefore, the **area is proportional to the square of x** .

If sales revenue is assumed to be proportional to the store area, we obtain:

$$f(x) = bx^2$$

This is an example of a **quadratic function**.

Quadratic functions and parabolas

The simplest quadratic function has the form:

$$f(x) = ax^2, \quad a \neq 0$$

Its graph is a *parabola* with vertex at the origin and symmetry axis equal to the y-axis.

More generally, a quadratic function has the form:

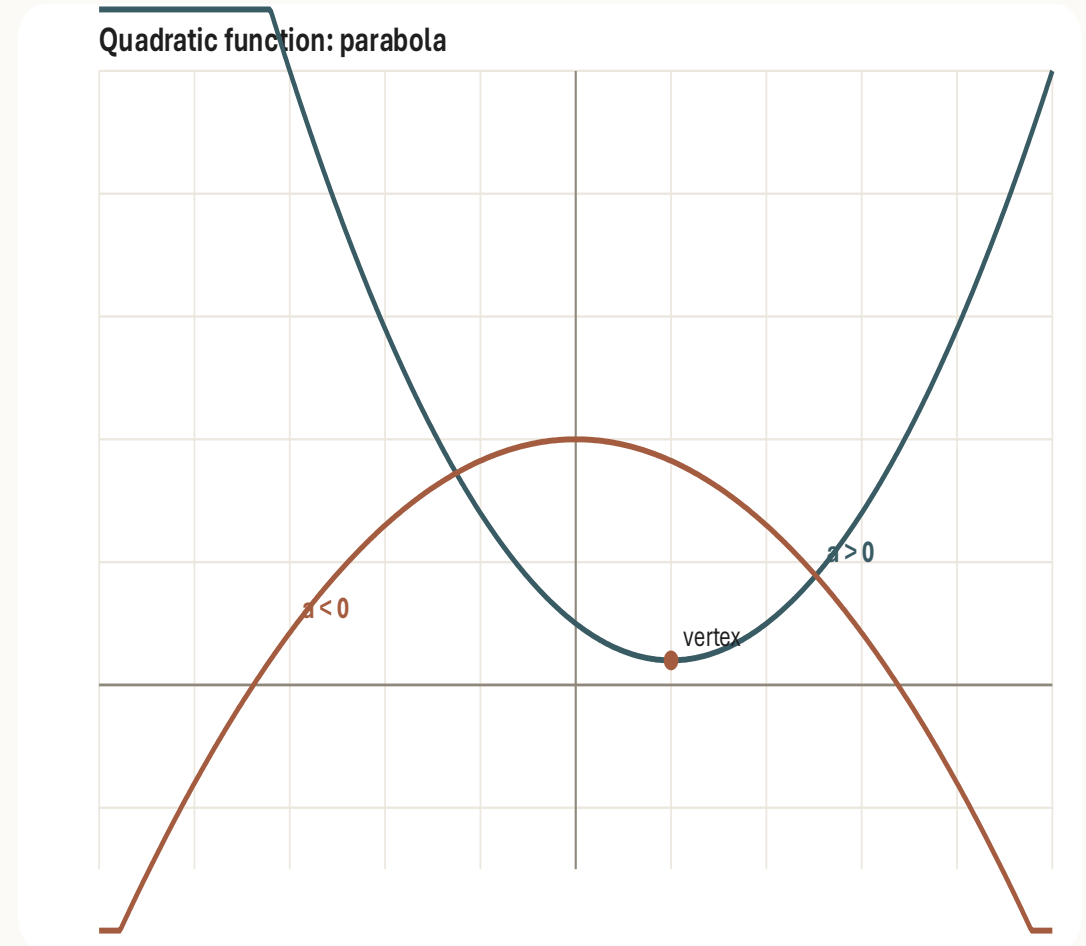
$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

Its axis of symmetry is:

$$x = -b / (2a)$$

The vertex is:

$$V = (-b / (2a), (4ac - b^2) / (4a))$$



Quadratic profit

Economic Application: Profit

A quadratic function can be used to model a firm's profit as a function of production:

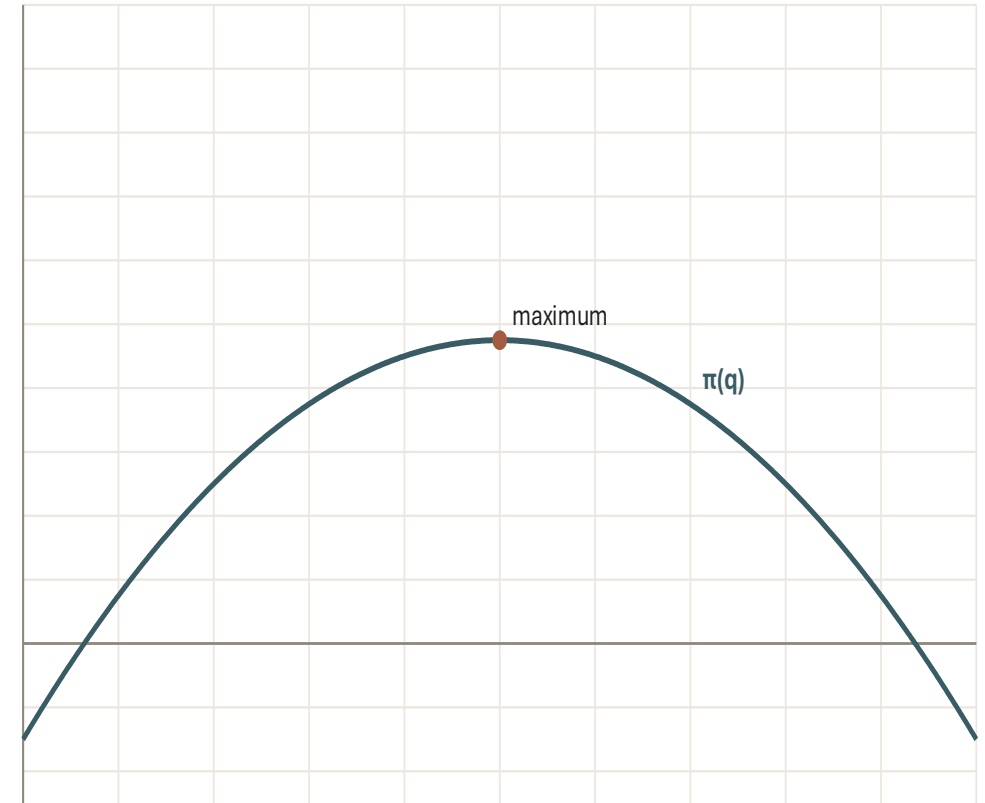
$$\pi(q) = aq^2 + bq + c$$

with:

$$a < 0, \quad b > 0, \quad c < 0$$

At low production levels, profit may be negative because of **fixed costs**. As production increases, profit rises until it reaches an **optimal production level**. Beyond this point, profit begins to decrease.

Monopoly profit as a concave quadratic



Inverse proportionality

Two non-zero variables x and y are inversely proportional when their product is constant.

$$xy = k, \quad k \neq 0$$

The corresponding function is:

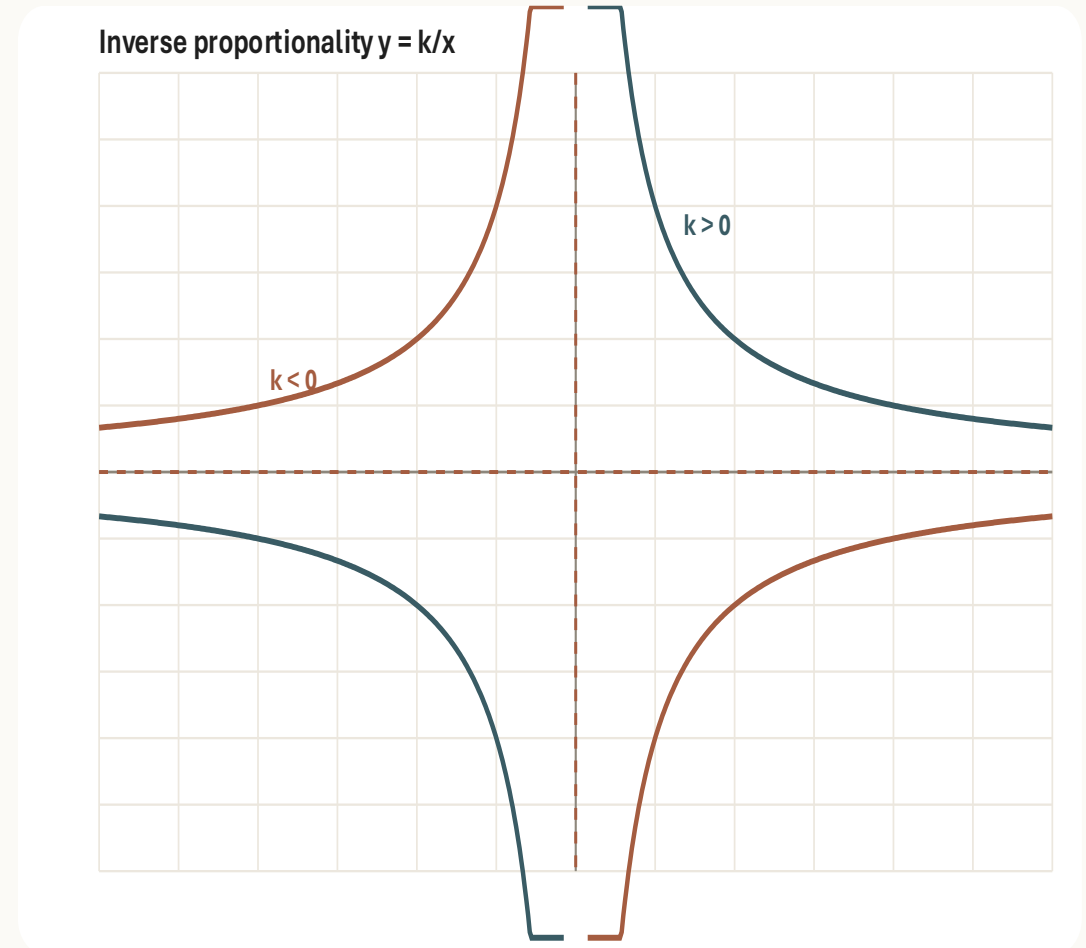
$$f(x) = k / x, \quad x \neq 0$$

Its graph is a hyperbola. The x -axis and y -axis are asymptotes.

As $|x|$ becomes very large:

$$f(x) \rightarrow 0$$

As x approaches 0, $|f(x)|$ becomes very large.



Economic example- Demand function

Let p be the price of a good and $q(p)$ the quantity demanded.

An inverse-proportional demand function has the form:

$$q(p) = k / p, \quad k > 0$$

As the price increases, the quantity demanded decreases.

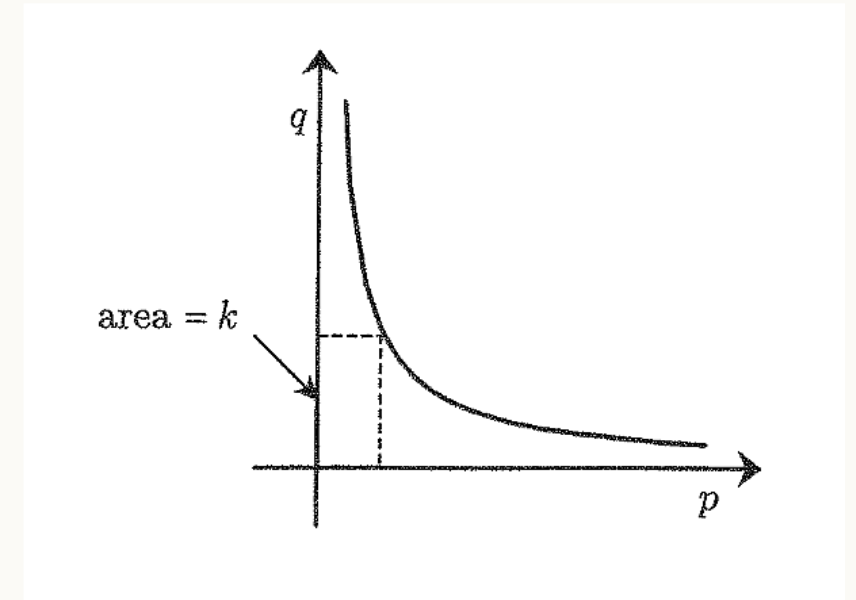
Total revenue is:

$$R(p) = p \cdot q(p)$$

Therefore:

$$R(p) = p \cdot (k / p) = k$$

So total remains constant at every price level.

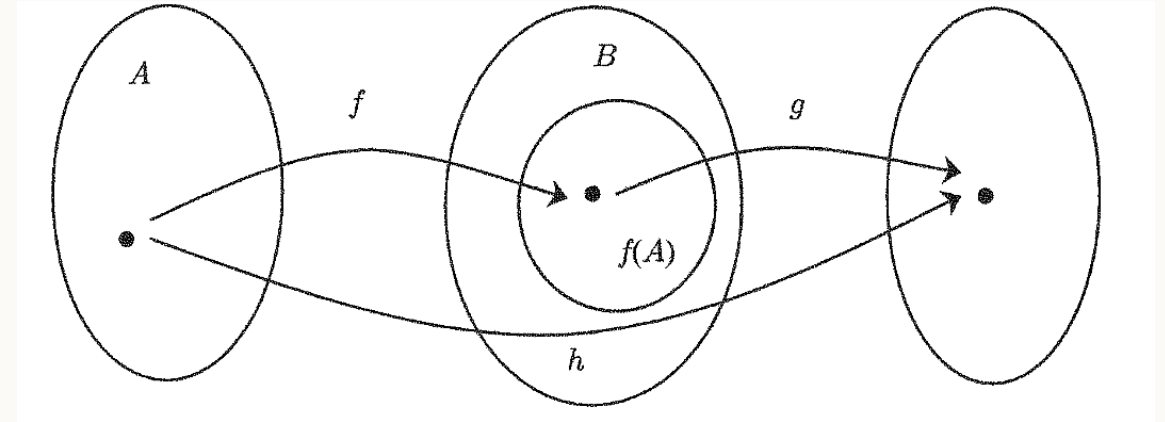


Function composition: motivation

Consider two functions:

$$f: A \rightarrow \mathbb{R}$$

$$g: B \rightarrow \mathbb{R}$$



and suppose that every value produced by f belongs to the domain of g :

$$f(A) \subseteq B$$

For each $x \in A$, the function f gives the value $f(x)$. Then g is applied to this result:

$$y = g(f(x))$$

This defines a new function $h: A \rightarrow \mathbb{R}$ called the *composition of f and g* .

$$h(x) = (g \circ f)(x) = g(f(x))$$

The symbol $g \circ f$ is read as “ g composed with f ”.

Composite function: example

Let:

$$f(x) = x + 3$$

$$g(x) = x^2$$

Since both functions are defined on $\mathbb{R}$, we can form both compositions.

First:

$$(g \circ f)(x) = g(f(x)) = g(x+3) = (x+3)^2$$

Now reversing the order:

$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 3$$

Therefore: $(g \circ f)(x) \neq (f \circ g)(x) !!!$

So, function **composition is generally not commutative.** In some cases, $g \circ f$ may be defined while $f \circ g$ is not.

From inverse relation to inverse function

A function is invertible when each output corresponds to exactly one input.

Definition

A function $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is invertible if and only if it is a bijective correspondence between A and $f(A)$.

The inverse function maps each $y \in f(A)$ to the unique x such that $f(x) = y$.

Main properties:

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

Graphically, a function is invertible if every horizontal line intersects its graph at most once. **The graphs of $y = f(x)$**

And $y = f^{-1}(x)$ are symmetric with respect to $y = x$

Example

$$f(x) = 3x - 4$$

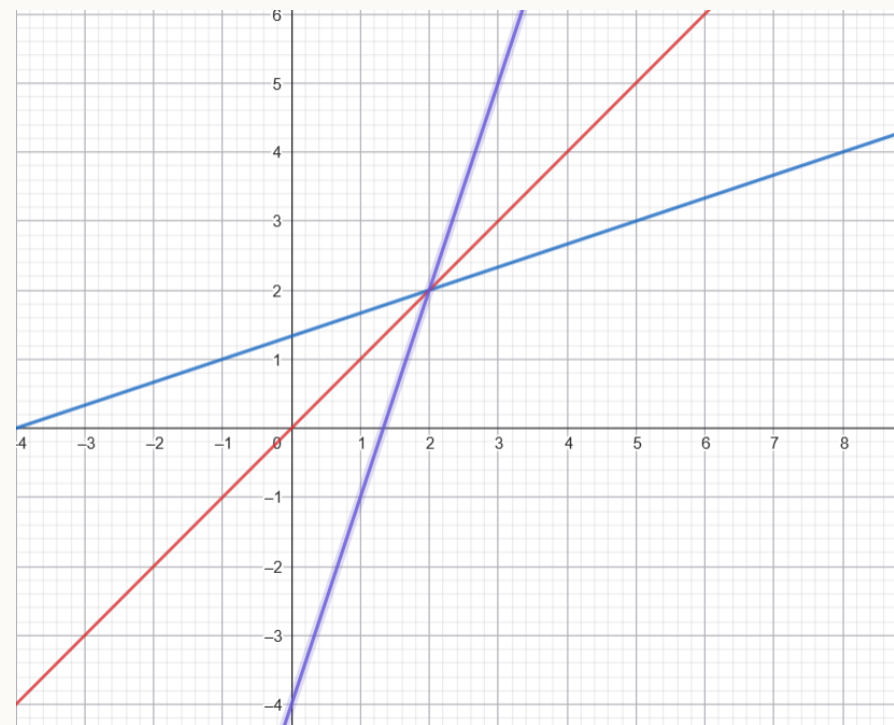
$$y = 3x - 4$$

$$x = (y + 4) / 3$$

Therefore:

$$f^{-1}(x) = (x + 4) / 3$$

An inverse function may exist even when no simple explicit formula can be found.



Bounded functions

A function f defined on a domain A is said to be *bounded above* if there exists a real number K such that:

$$f(x) \leq K \text{ for every } x \in A.$$

Graphically, the entire graph of f lies below the horizontal line $y = K$.

A function *is bounded below* if there exists a real number H such that:

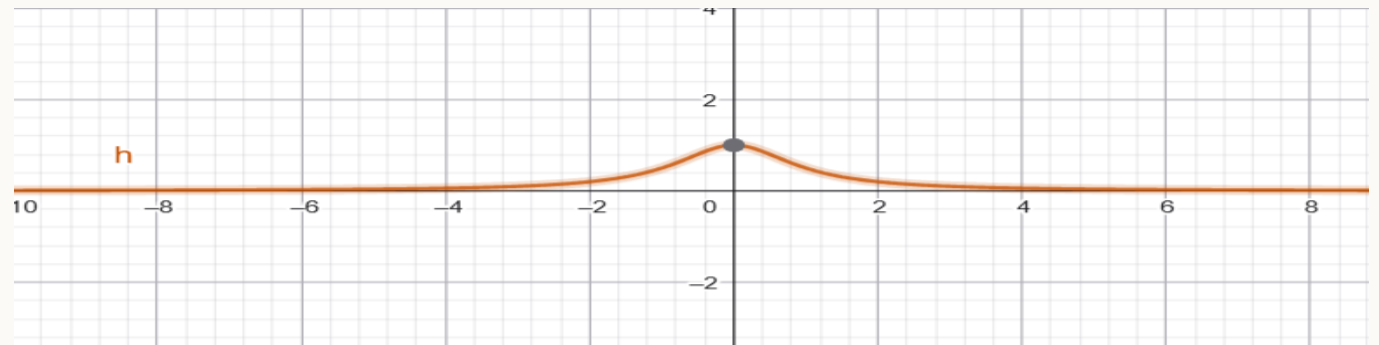
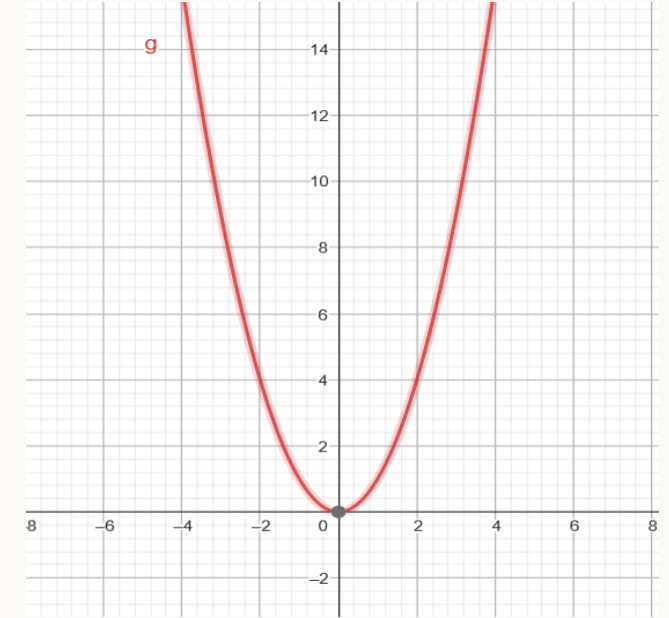
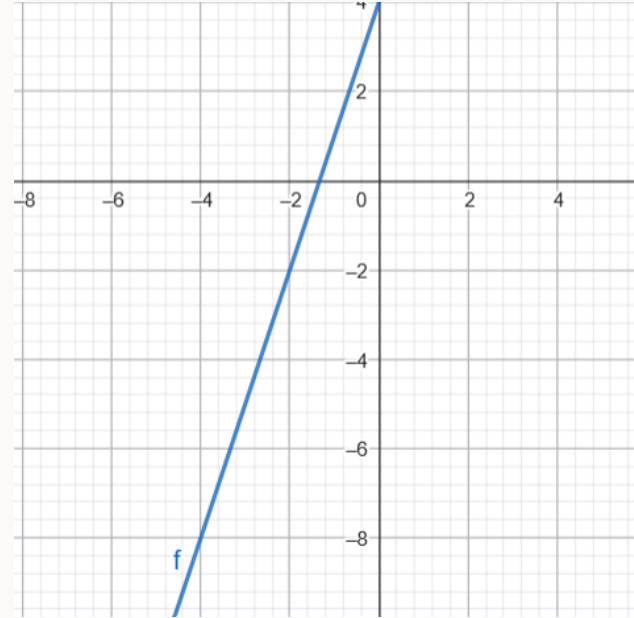
$$f(x) \geq H \text{ for every } x \in A.$$

Graphically, the entire graph of f lies above the horizontal line $y = H$.

A function that is both bounded above and bounded below is called a *bounded function*.

Examples

- $f(x) = 3x + 4$. The function f is neither bounded above nor bounded below.
- $g(x) = x^2$. The function g is bounded below by 0: $g(x) \geq 0$ but it is not bounded above.
- $h(x) = 1 / (1 + x^2)$. The function h is bounded because, for every $x \in \mathbb{R}$: $0 < h(x) \leq 1$. Therefore, 0 is a lower bound and 1 is an upper bound.



Monotone functions

Definition – Monotone Functions

A function $f: A \rightarrow \mathbb{R}$ is called non-decreasing if, for every $x_1, x_2 \in A$:

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

A function is called non-increasing if:

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

If the inequalities are strict, the function is called *strictly increasing* or *strictly decreasing*.

Increasing and decreasing functions are collectively called monotone functions.

Definition – Monotone Sequences

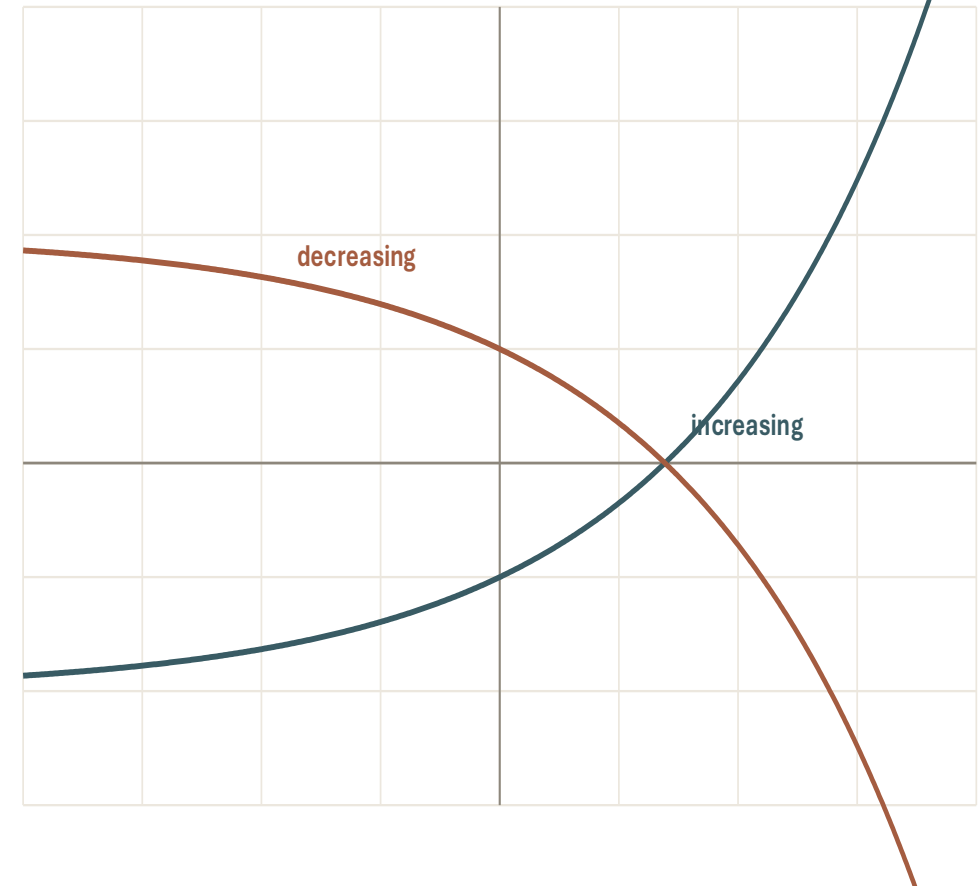
A sequence $\{a_n\}$ is called non-decreasing if:

$$a_{n+1} \geq a_n$$

It is called non-increasing if:

$$a_{n+1} \leq a_n$$

Increasing versus decreasing



Example— Geometric sequences by q

Consider the geometric sequence:

$$a_n = q^n$$

Its behavior depends on the value of q .

- If $q > 1$, the sequence is strictly increasing:

$$1, 2, 4, 8, \dots, 2^n, \dots$$

- If $q = 1$, the sequence is constant:

$$1, 1, 1, 1, \dots$$

- If $0 < q < 1$, the sequence is strictly decreasing:

$$1, 1/2, 1/4, 1/8, \dots, 1/2^n, \dots$$

- If $q = 0$, the sequence is constant from $n \geq 1$.
- If $q < 0$, the terms alternate in sign, so the sequence is neither increasing nor decreasing.

Global maxima and minima

Definition – Global Maximum

Let f be a function defined on a domain A . A real number M is called the global (or absolute) maximum of f if there exists $x_0 \in A$ such that:

$$M = f(x_0) \geq f(x) \text{ for every } x \in A.$$

The point x_0 is called a global maximum point.

Definition – Global Minimum

A real number m is called the global (or absolute) minimum of f if there exists $x_0 \in A$ such that:

$$m = f(x_0) \leq f(x) \text{ for every } x \in A.$$

The point x_0 is called a global minimum point.

- A maximum or minimum point is called strong if equality occurs only at $x = x_0$.
- Maximum and minimum points are also called extremum points, while their values are called extrema.
- If they exist, the maximum and minimum values are unique, but a function may have several points where the same maximum or minimum is attained.

Convex and Concave functions

Definition – Convex Set: A set is called convex if the line segment joining any two of its points lies entirely inside the set.

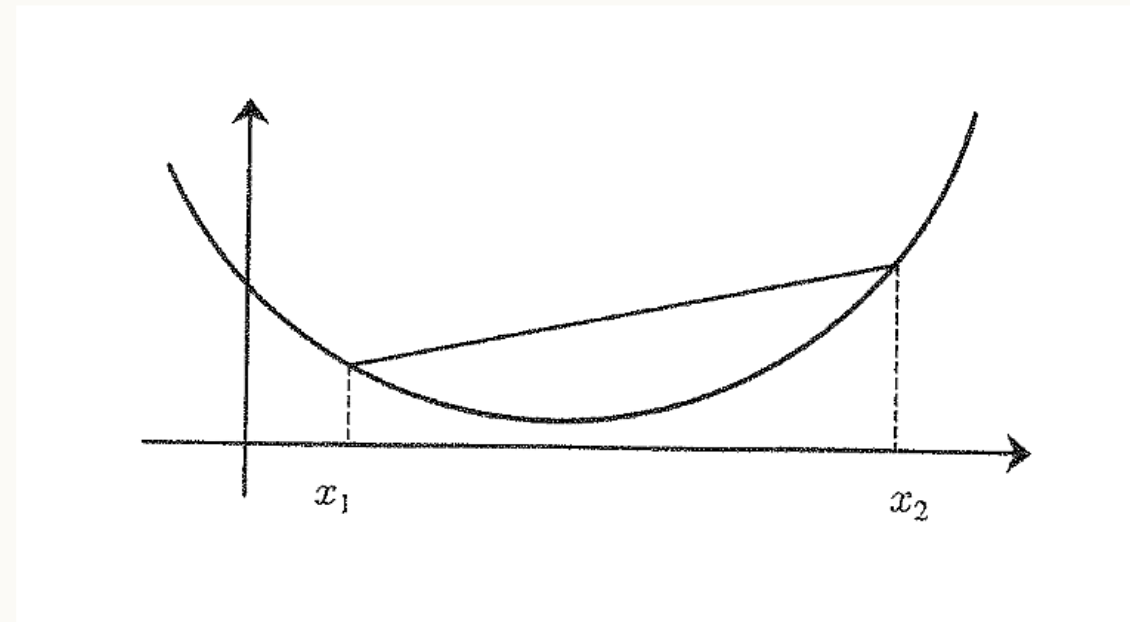
Typical examples are disks, ellipses, and half-planes.

Definition – Convex Function: A function f defined on an interval is convex if the segment joining any two points of its graph lies above, or on, the graph. Equivalently, for every x_1, x_2 in the domain and every $t \in [0, 1]$:

$$f(tx_1 + (1 - t)x_2) \leq t f(x_1) + (1 - t) f(x_2)$$

If the inequality is strict whenever $x_1 \neq x_2$ and $0 < t < 1$, the function is strictly convex.

For example $f(x) = ax^2$, $a > 0$ is strictly convex.



Definition – Concave Function: A function f is concave if $-f$ is convex.

Equivalently:

$$f(tx_1 + (1 - t)x_2) \geq t f(x_1) + (1 - t) f(x_2)$$

If the inequality is strict, the function is strictly concave.

For example $f(x) = ax^2$, $a < 0$ is strictly concave.

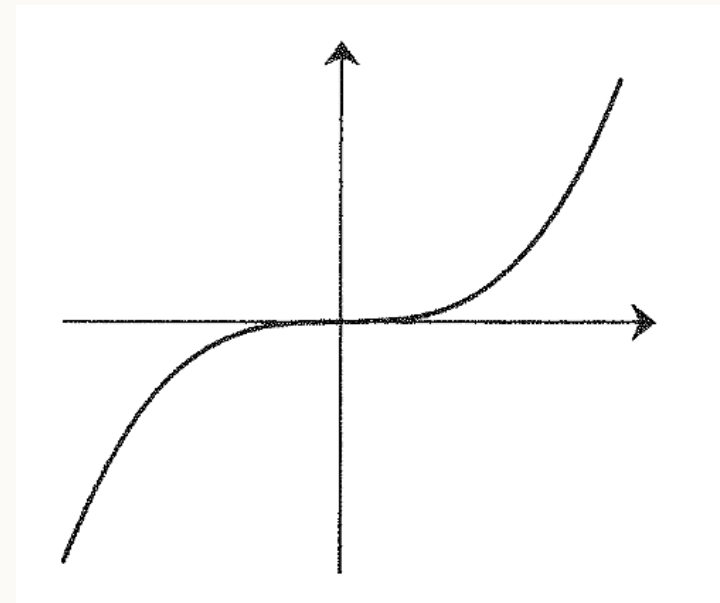
Definition – Inflection Point: A point x_0 is called an inflection point if the function changes its concavity around x_0 (convex $\rightarrow$ concave or concave $\rightarrow$ convex)

A typical example is:

$$f(x) = x^3$$

which has an inflection point at:

$$x = 0$$



Local properties

Sometimes we are interested in the **behavior** of a function only **near a specific point of its domain**. These are called **local properties**, as opposed to global properties, which hold over the entire domain.

For example, a function may be increasing near a point even if it is not increasing on its whole domain.

Definition – Local Maximum: Let x_0 belong to the domain of f . The point x_0 is called a **local maximum point** if there exists a neighborhood of $x_0 : (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that:

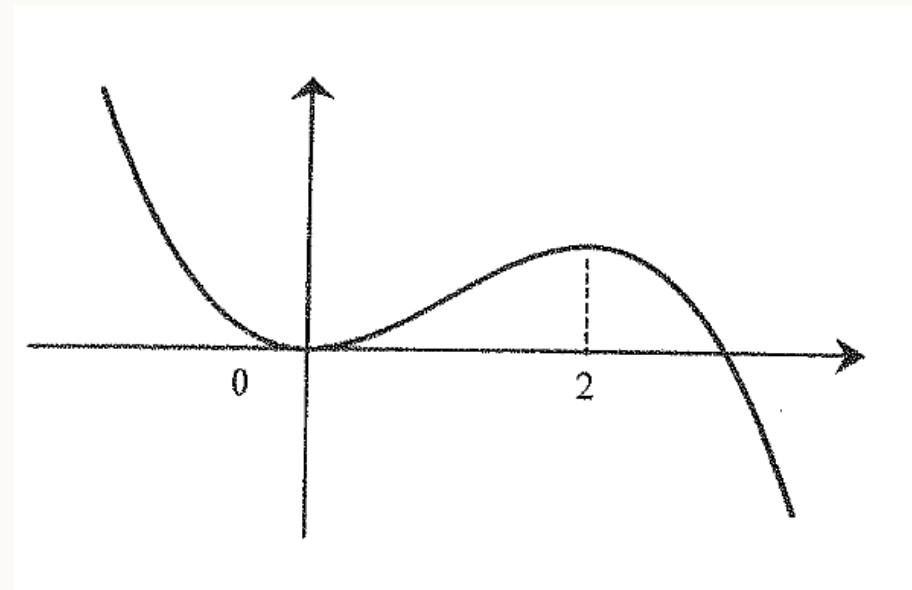
$$f(x_0) \geq f(x)$$

for every x in that neighborhood and in the domain of f .

Definition – Local Minimum: The point x_0 is called a **local minimum point** if there exists a neighborhood of $x_0 : (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that:

$$f(x_0) \leq f(x)$$

for every x in that neighborhood and in the domain of f .



Local extrema

Strong Local Extrema

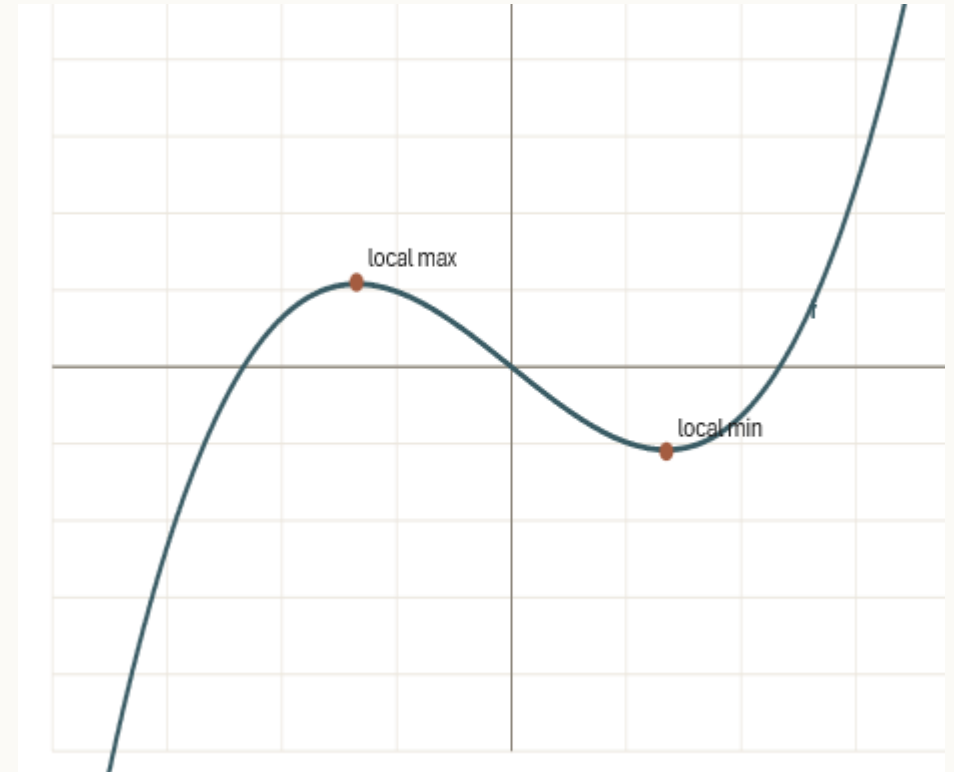
A local maximum or minimum is called strong if equality holds only when:

$$x = x_0$$

Local vs Global Properties

Every global maximum or minimum is also a local maximum or minimum.

However, the converse is not necessarily true: **a function may have a local extremum that is not a global extremum.**



Power functions

The functions previously considered, such as linear, quadratic and inverse proportionality functions, are special cases of power functions.

A power function has the form:

$$f(x) = kx^{\alpha}, \quad k \neq 0$$

where α is a real number.

In general, power functions are defined for $x > 0$. If $\alpha > 0$, the function can also be defined at $x = 0$, with $f(0) = 0$.

Some particular cases allow the function to be defined on a larger domain.

- $f(x) = x^2$ and $f(x) = x^3$ are defined on $\mathbb{R}$
- $f(x) = 1/x = x^{-1}$ and $f(x) = 1/x^2 = x^{-2}$ are defined on $\mathbb{R} \setminus \{0\}$
- $f(x) = \sqrt{x} = x^{1/2}$ is defined for $x \geq 0$

Main Properties

For $k > 0$:

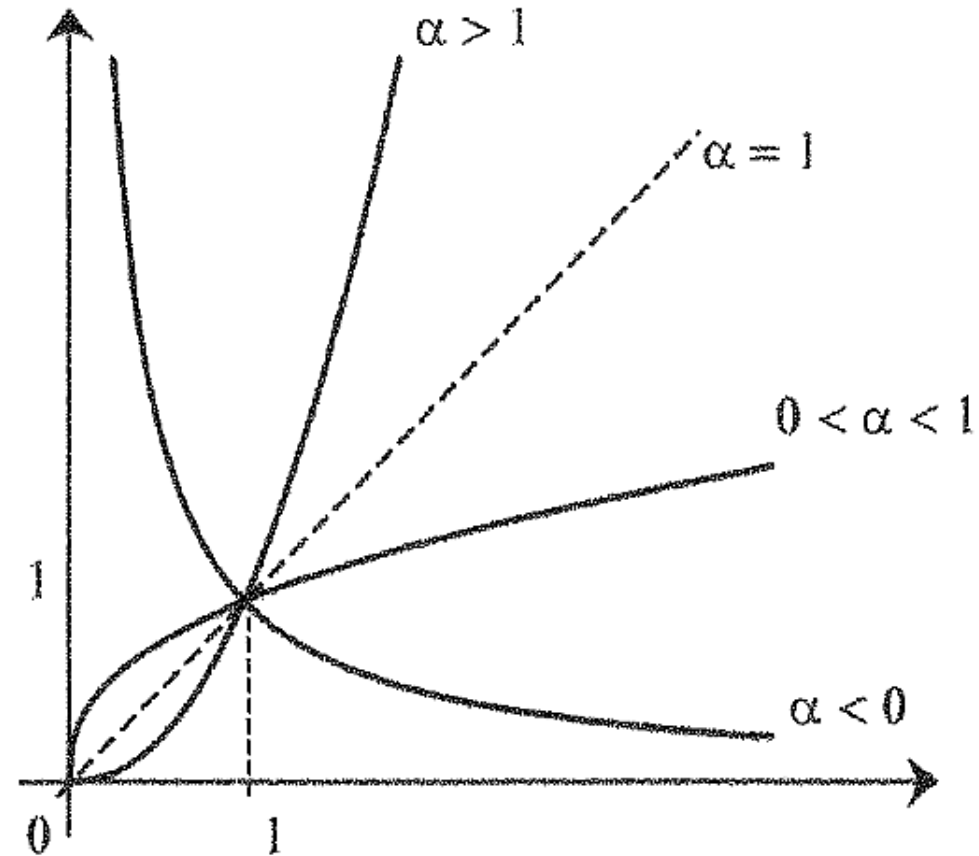
- $\alpha > 0 \rightarrow$ strictly increasing
- $\alpha < 0 \rightarrow$ strictly decreasing
- $\alpha < 0$ or $\alpha > 1 \rightarrow$ strictly convex
- $0 < \alpha < 1 \rightarrow$ strictly concave

Behavior at Infinity

- If $\alpha > 0$: $x^\alpha \rightarrow +\infty$ as $x \rightarrow +\infty$
- If $\alpha < 0$: $x^\alpha \rightarrow 0$ as $x \rightarrow +\infty$

For negative exponents:

$$x^\alpha \rightarrow +\infty \text{ as } x \rightarrow 0^+$$



Example — Inventory management

A company wants to determine the optimal order quantity Q for purchasing raw materials. The optimal quantity is:

$$Q = \sqrt{(2gs / m)}$$

where:

s = annual demand for raw materials

g = fixed cost of placing an order

m = annual holding cost per unit of inventory

This formula is an example of a power function.

If Q is considered as a function of s or g :

$Q \propto s^{1/2}$ and $Q \propto g^{1/2}$ so the exponent is: $\alpha = 1/2$

If Q is considered as a function of m : $Q \propto m^{-1/2}$ so the exponent is:
 $\alpha = -1/2$

Even and odd functions

Some functions have **special symmetries** in their graphs.

Definition – Even Function: A function f is called **even** if:

$$f(-x) = f(x)$$

for every x in its domain.

Its graph is symmetric with respect to the y-axis.

Examples: $f(x) = x^2$, $f(x) = x^4$.

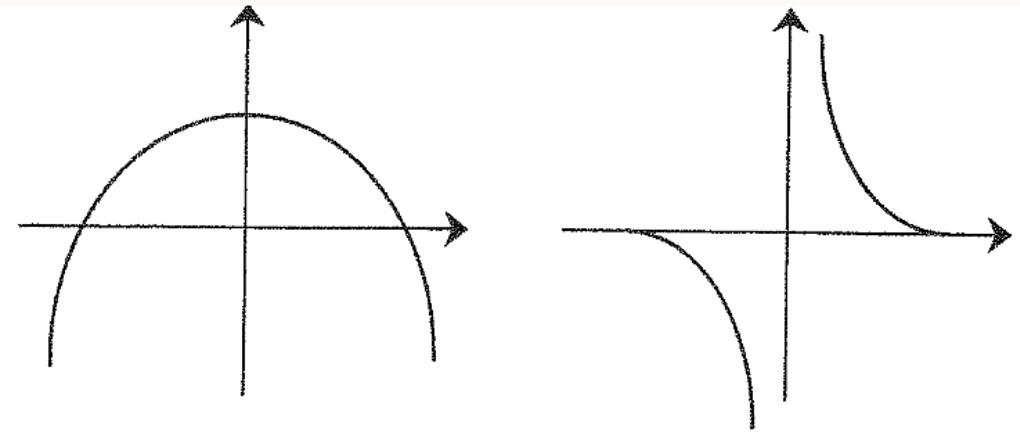
Definition – Odd Function: A function f is called **odd** if:

$$f(-x) = -f(x)$$

for every x in its domain.

Its graph is symmetric with respect to the origin.

Example: $f(x) = x^3$



EVEN

ODD

Invertibility of power functions

Some power functions are invertible, while others become invertible only after restricting their domain.

- The function $f(x) = x$ is its own inverse: $f^{-1}(x) = x$
- The function $f(x) = 1/x$ is also its own inverse: $f^{-1}(x) = 1/x$
- The function $f(x) = x^3$ is invertible on $\mathbb{R}$, and its inverse is: $f^{-1}(x) = \sqrt[3]{x}$
- The function $f(x) = x^2$ is not invertible on $\mathbb{R}$ because different inputs can give the same output: $f(x) = f(-x)$. However, restricting the domain to $x \geq 0$ makes it invertible. In this case: $f^{-1}(x) = \sqrt{x}$
- For $\alpha \neq 0$, the power function $f(x) = x^\alpha$, $x > 0$ is invertible, and its inverse is another power function: $f^{-1}(x) = x^{1/\alpha}$

Important

Do not confuse the inverse function $f^{-1}(x)$ with the reciprocal $1/f(x)$!

For example, if $f(x) = x^2$ then:

$$f^{-1}(x) = \sqrt{x}$$

while:

$$1/f(x) = 1/x^2$$

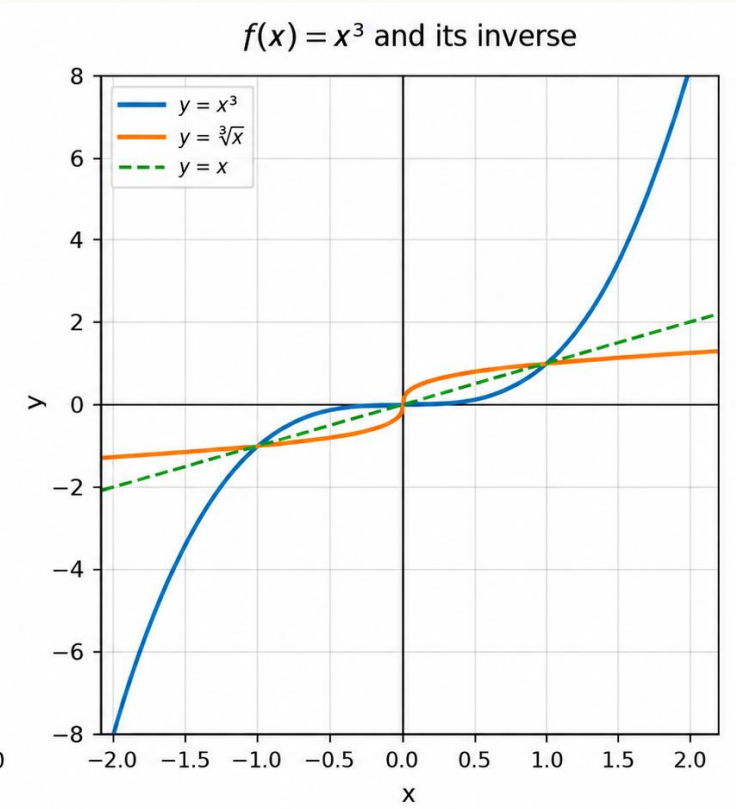
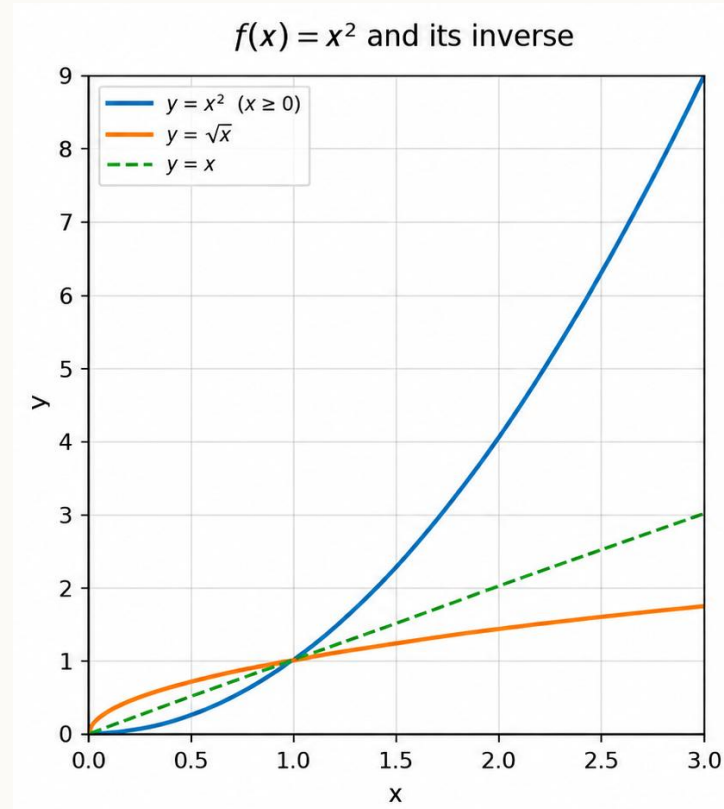
Example – Finding the Inverse Function

To find the inverse of a simple function:

1. Write the function as $y = f(x)$.
2. Solve the equation for x .
3. Exchange x and y .

Examples

- $f(x) = x^2, x \geq 0 \rightarrow$ Write: $y = x^2 \rightarrow$ Solve for x : $x = \sqrt{y} \rightarrow$ Therefore: $f^{-1}(x) = \sqrt{x}$
- $f(x) = x^3 \rightarrow$ Write: $y = x^3 \rightarrow$ Solve for x : $x = \sqrt[3]{y} \rightarrow$ Therefore: $f^{-1}(x) = \sqrt[3]{x}$



Exponential functions

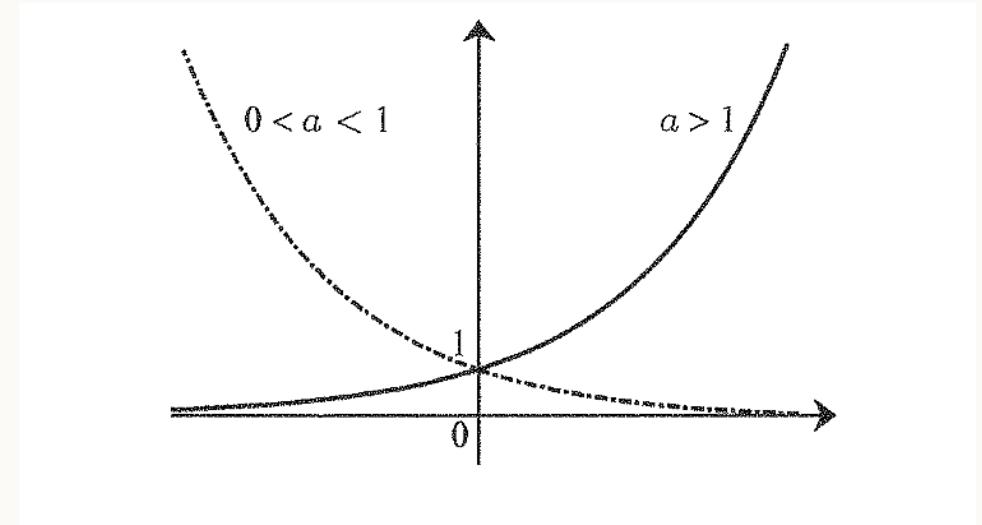
Exponential functions are widely used in economic models to describe phenomena such as depreciation of machinery, diffusion of a new product, market-share growth:

$$f(x) = a^x$$

where: $a > 0$, $a \neq 1$.

Main Properties

- Exponential functions are always positive: $f(x) > 0$
- Their graph always passes through $(0, 1)$
- If $a > 1$, the function is strictly increasing.
- If $0 < a < 1$, the function is strictly decreasing.
- For every admissible value of a , the function is strictly convex.
- If $a > 1$: $a^x \rightarrow +\infty$ as $x \rightarrow +\infty$
- If $0 < a < 1$: $a^x \rightarrow 0$ as $x \rightarrow +\infty$
- **Symmetry.** Two exponential functions with reciprocal bases: $f(x) = a^x$ and $g(x) = (1/a)^x$ have graphs that are symmetric with respect to the y-axis.



Logarithmic functions

Every exponential function is invertible. The inverse of

$$f(x) = a^x, \quad a > 0, \quad a \neq 1$$

is called the logarithmic function with base a :

$$f^{-1}(x) = \log_a(x)$$

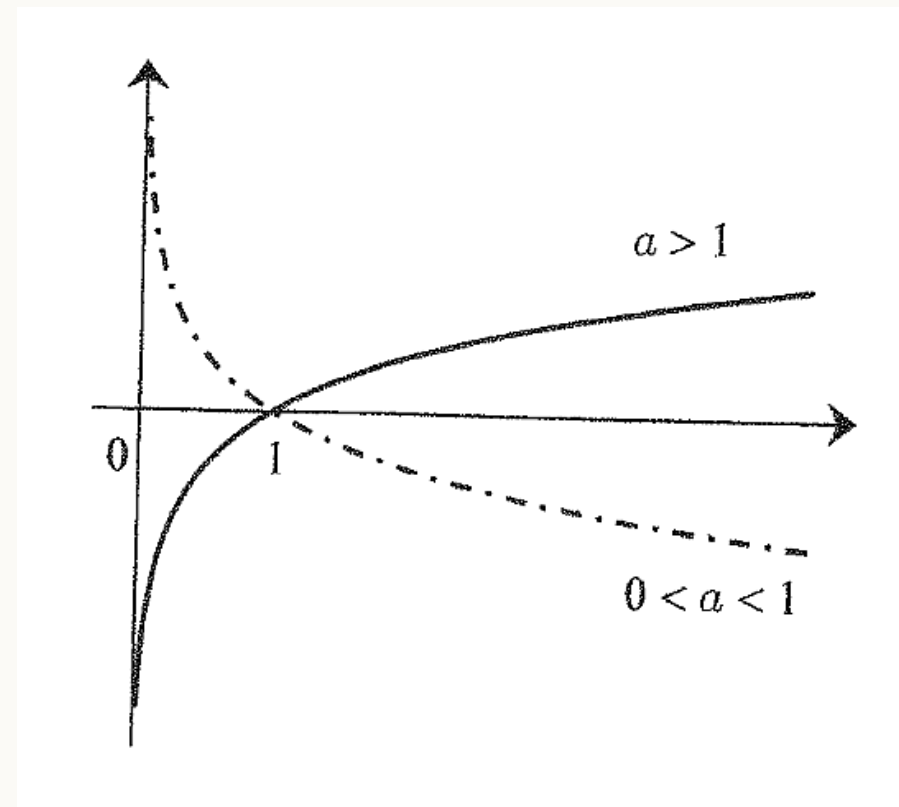
- The domain of a logarithmic function is: $(0, +\infty)$
- The graphs of $y = a^x$ and $y = \log_a(x)$ are symmetric with respect to the line: $y = x$

Main Properties

- If $a > 1$ $\log_a(x)$ is strictly increasing and concave.
- If $0 < a < 1$ $\log_a(x)$ is strictly decreasing and convex.
- For every admissible base: $\log_a(1) = 0$.
- If $a > 1$: $\log_a(x) \rightarrow -\infty$ as $x \rightarrow 0^+$ and $\log_a(x) \rightarrow +\infty$ as $x \rightarrow +\infty$
- If $0 < a < 1$: $\log_a(x) \rightarrow +\infty$ as $x \rightarrow 0^+$ and $\log_a(x) \rightarrow -\infty$ as $x \rightarrow +\infty$

The logarithm with base e is called the natural logarithm:

$$\ln(x) = \log_e(x)$$



Trigonometric functions

Trigonometric functions are useful for describing **periodic phenomena**, such as waves, seasonal patterns and **cyclical economic behavior**.

The main trigonometric functions are **sine, cosine and tangent**.

Sine and Cosine

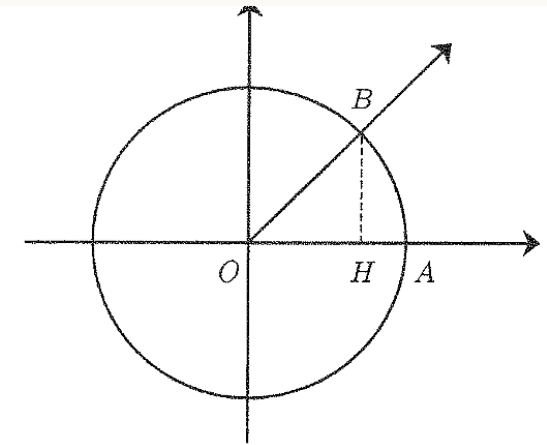
Using the unit circle, $A=(1,0)$, an angle x measured in radians:

- $\sin(x)$ = y-coordinate of the corresponding point on the unit circle
- $\cos(x)$ = x-coordinate of the corresponding point

Both functions are periodic with period 2π :

- $\sin(x + 2\pi) = \sin(x)$
- $\cos(x + 2\pi) = \cos(x)$

Definition – Periodic Function: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is periodic with period T if $T > 0$ is the smallest positive number such that: **$f(x + T) = f(x)$** for every x in the domain.



Main Properties

- Fundamental identity: $\cos^2(x) + \sin^2(x) = 1$
- Symmetry:
 $\sin(-x) = -\sin(x)$ odd
 $\cos(-x) = \cos(x)$ even
- Boundedness:

$$-1 \leq \sin(x) \leq 1$$

$$-1 \leq \cos(x) \leq 1$$

- Translation Formulas
 $\cos(x) = \sin(x + \pi/2)$
 $\sin(x) = \cos(x - \pi/2)$
- Addition and Subtraction Formulas
 $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
 $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
 $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
 $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

Tangent function

The tangent function is defined by:

$$\tan(x) = \sin(x) / \cos(x)$$

It is not defined when:

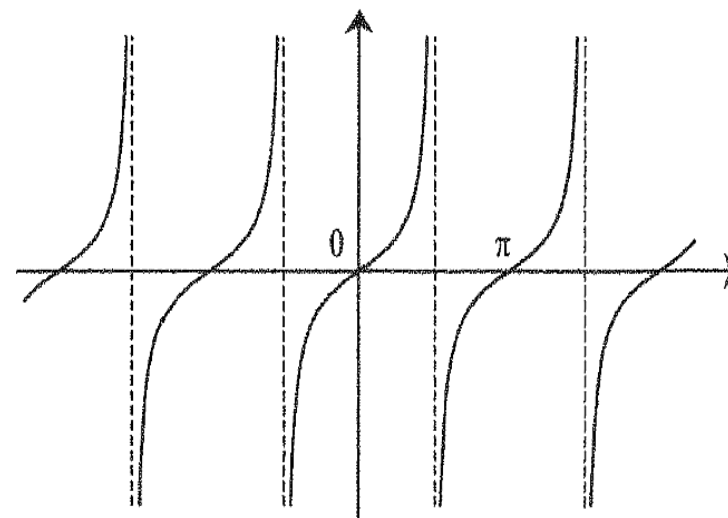
$$x = \pi/2 + k\pi, \quad k \in \mathbb{Z}$$

The tangent function is **odd**:

$$\tan(-x) = -\tan(x)$$

and periodic with period π :

$$\tan(x + \pi) = \tan(x)$$



Arctangent

Because tangent is periodic, it is not invertible on its whole domain.

Restricting tangent to:

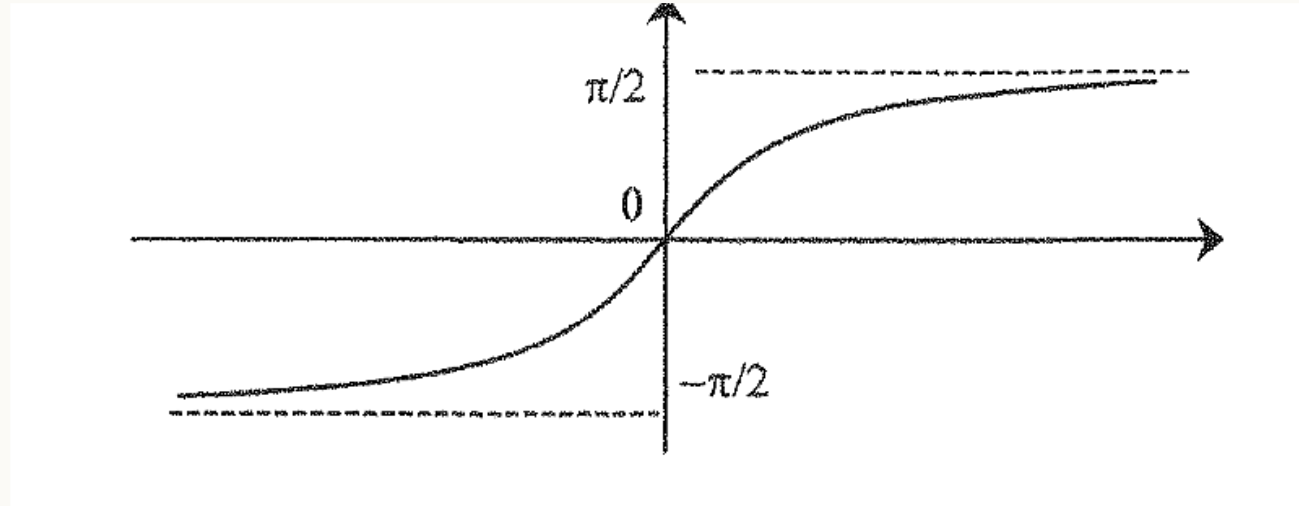
$$(-\pi/2, \pi/2)$$

makes it invertible.

Its inverse is the arctangent function:

$$y = \arctan(x)$$

The function $\arctan(x)$ is: strictly increasing, odd, bounded between $-\pi/2$ and $\pi/2$ with:
 $-\pi/2 < \arctan(x) < \pi/2$



Geometric transformations

Vertical and Horizontal Translations

Starting from a function $f(x)$: **$f(x) + k$ shifts the graph vertically.**

- $k > 0 \rightarrow$ upward by k units
- $k < 0 \rightarrow$ downward by $|k|$ units

Instead: **$f(x + k)$ shifts the graph horizontally.**

- $k > 0 \rightarrow$ left by k units
- $k < 0 \rightarrow$ right by $|k|$ units

Homographic Functions

A homographic function has the form: $f(x) = (ax + b) / (x + c)$

Its graph is a translated hyperbola. The asymptotes are: $x = -c$, $y = a$.

Example

$f(x) = 2x / (x - 1)$ can be rewritten as:

$$f(x) = 2 + 2 / (x - 1)$$

Therefore, the graph of $y = 2/x$ is shifted:

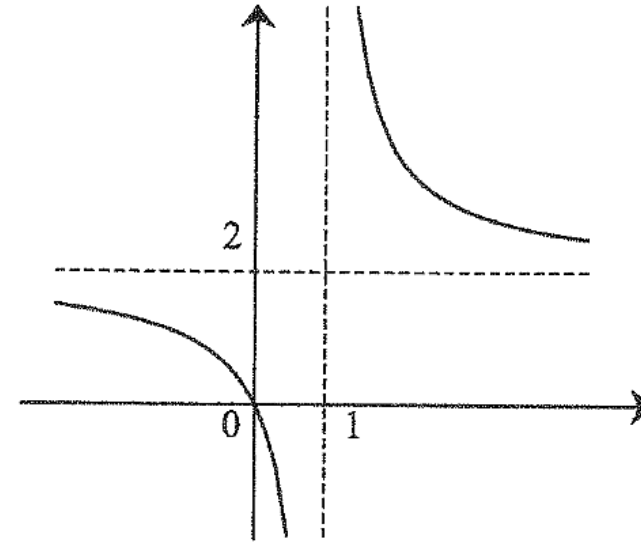
- 1 unit to the right
- 2 units upward

The asymptotes are: $x = 1$, $y = 2$

Local Change of Coordinates

When studying a function near a point x_0 , it is often useful to move the reference point to the origin.

Define: $g(h) = f(x_0 + h) - f(x_0)$ This transformation moves the point $(x_0, f(x_0)) \rightarrow (0,0)$



$$f(x) = 2 + 2/(x-1)$$

Vertical/Horizontal Scaling and Reflection

Multiplying a function by a constant k gives:

$$y = kf(x)$$

- If $k > 1 \rightarrow$ vertical stretching
- If $0 < k < 1 \rightarrow$ vertical compression
- If $k < 0 \rightarrow$ reflection across the x -axis, together with scaling. In particular: $y = -f(x)$ is the reflection of $y = f(x)$ across the x -axis.

Horizontal Scaling and Reflection

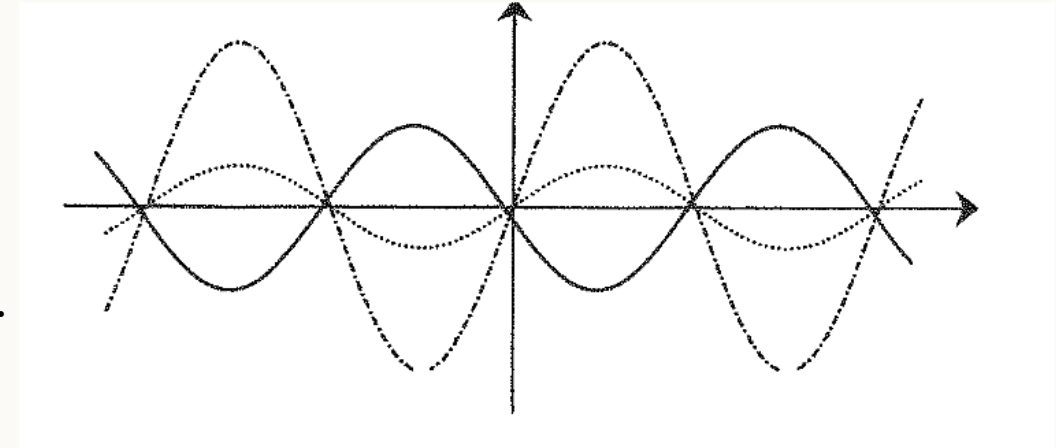
Replacing x with kx gives:

$$y = f(kx)$$

If $k > 1 \rightarrow$ horizontal compression

If $0 < k < 1 \rightarrow$ horizontal stretching

If $k < 0 \rightarrow$ reflection across the y -axis, together with scaling. In particular: $y = f(-x)$ is the reflection of $y = f(x)$ across the y -axis.



-----	$y = 3 \sin x$
-----	$y = \frac{1}{2} \sin x$
—————	$y = -\sin x$

Absolute value of a function

The absolute value function is:

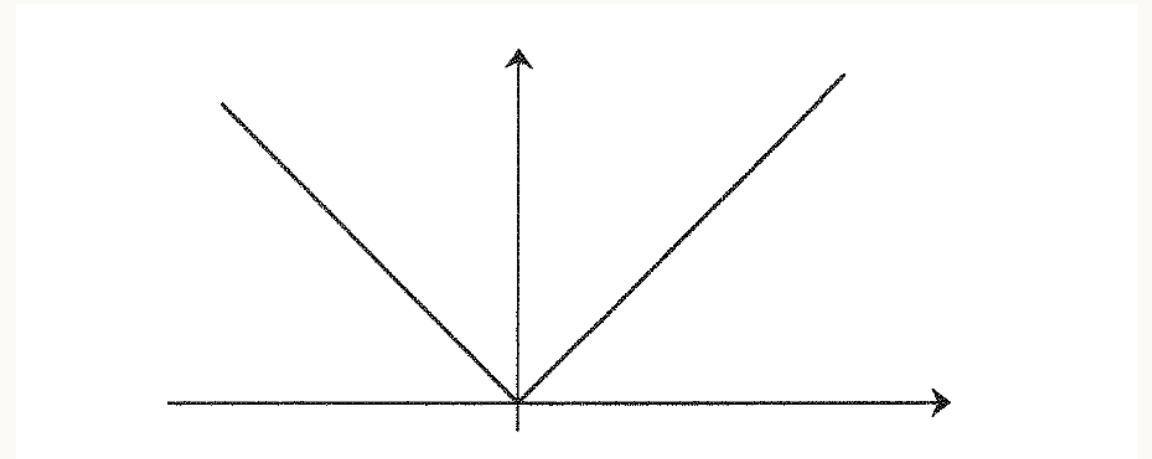
$$f(x) = |x|$$

Its graph has the characteristic V-shape.

For a general function f : $y = |f(x)|$ reflects every part of the graph below the x-axis upward.

Equivalently:

- $|f(x)| = f(x)$ if $f(x) \geq 0$
- $|f(x)| = -f(x)$ if $f(x) < 0$
- Instead: $y = f(|x|)$ keeps the right-hand part of the graph and reflects it across the y-axis. Therefore: $f(|x|)$ is always an even function.



$$f(x) = |x|$$

Take-Home Message

- A function assigns one unique output to every admissible input.
- Graphs translate algebraic relationships and economic models into geometric information.
- Sequences and elementary functions provide useful models for economic and mathematical phenomena.
- Key tools include linear and affine functions, quadratic models, inverse proportionality, composition, inverse functions, monotonicity, extrema, convexity and concavity.
- Exponential, logarithmic and trigonometric functions extend this toolkit to growth, decay and periodic phenomena.
- These concepts provide the foundation for the next chapter: limits.