Kinematics I — vectors: position, velocity, acceleration

- Vectors
- Position, displacement, velocity, and acceleration vectors
- Graphical relations: position–velocity and velocity–acceleration
- Examples

Direction matters: vectors in the living world

Many biological quantities have a direction, not just a size: a cell's motion, blood flow, a diffusing molecule, the pull in a centrifuge.

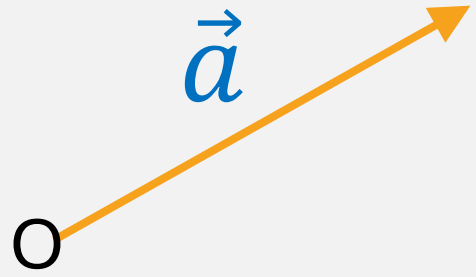
Vectors add tip-to-tail: a microbe swimming across a flow follows the sum of its own velocity and the flow's.

Velocity is the rate of change of position; acceleration is the rate of change of velocity — both are vectors.

Net displacement (start → end) is usually far shorter than the path actually travelled.

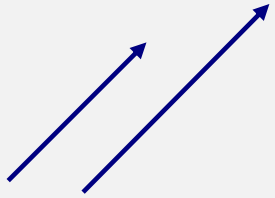
Spinning in a centrifuge means constant acceleration toward the axis — the basis of separation.

VECTORS

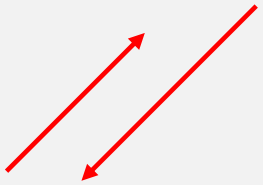


Vector $\vec{a}$ o $\boldsymbol{a}$

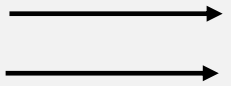
The magnitude of the vector is $|\overrightarrow{a}| = a$ and, graphically, it corresponds to the length of the arrow.
The origin of the vector is point O.



CONCORDANT
(parallel)
same direction and same sense



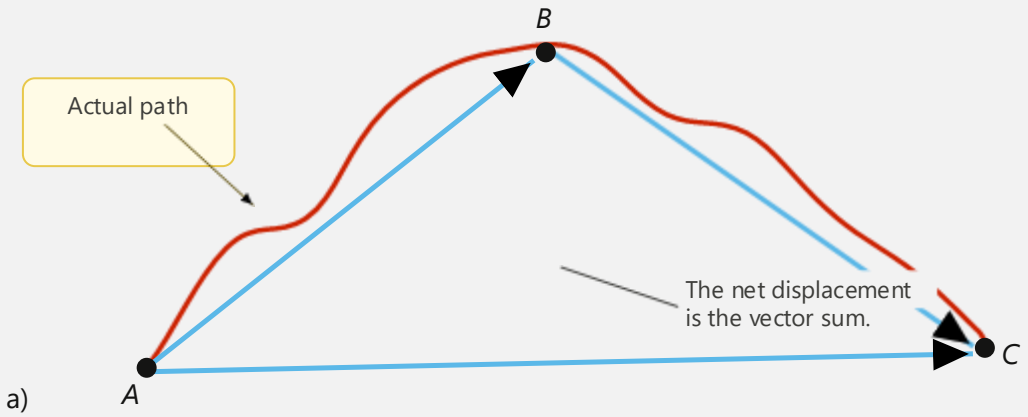
DISCORDANT
(anti-parallel)
same direction but opposite sense

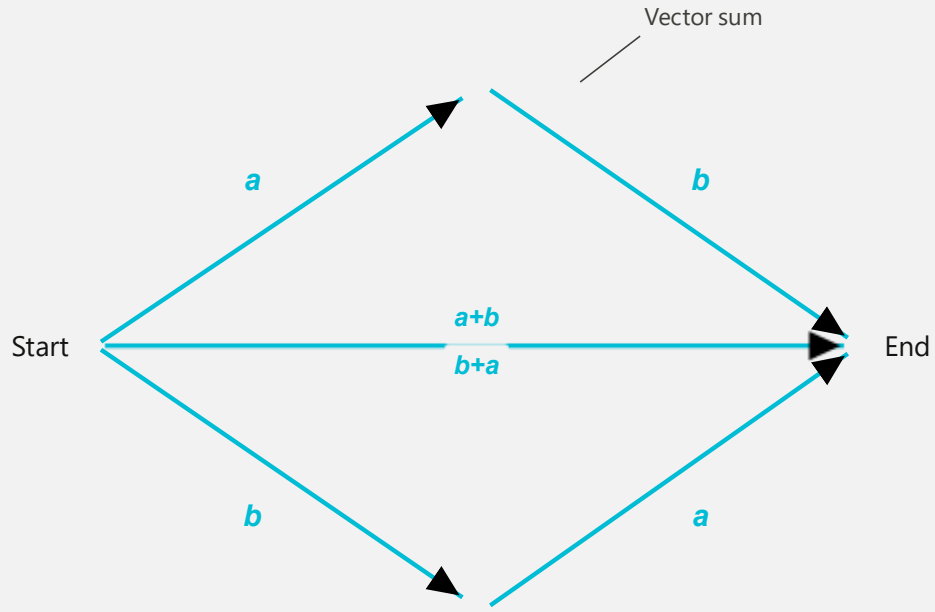


EQUIPOLLENT
Equal in magnitude, sense, and direction

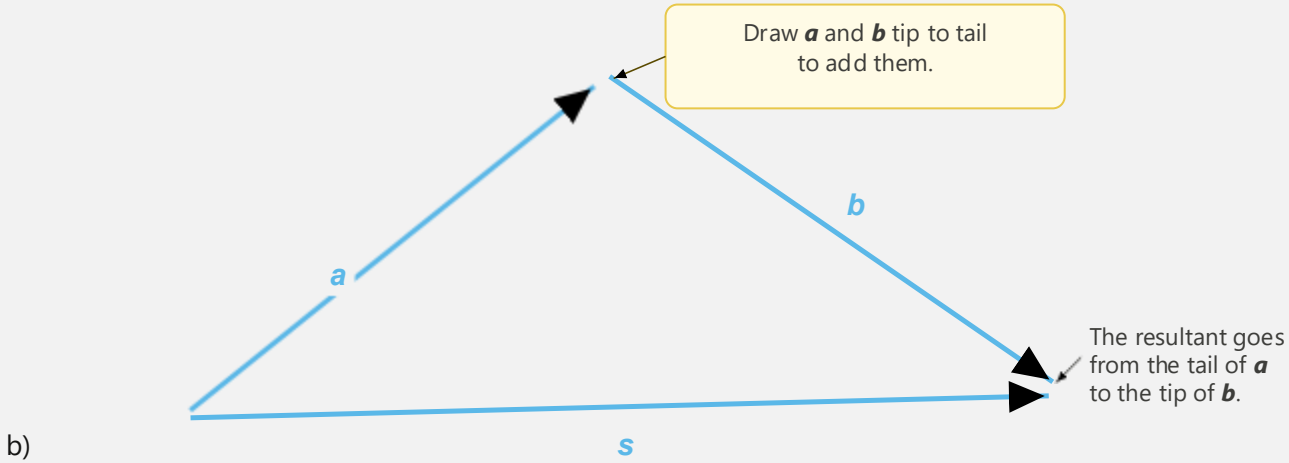
VECTOR OPERATIONS

Vector Addition



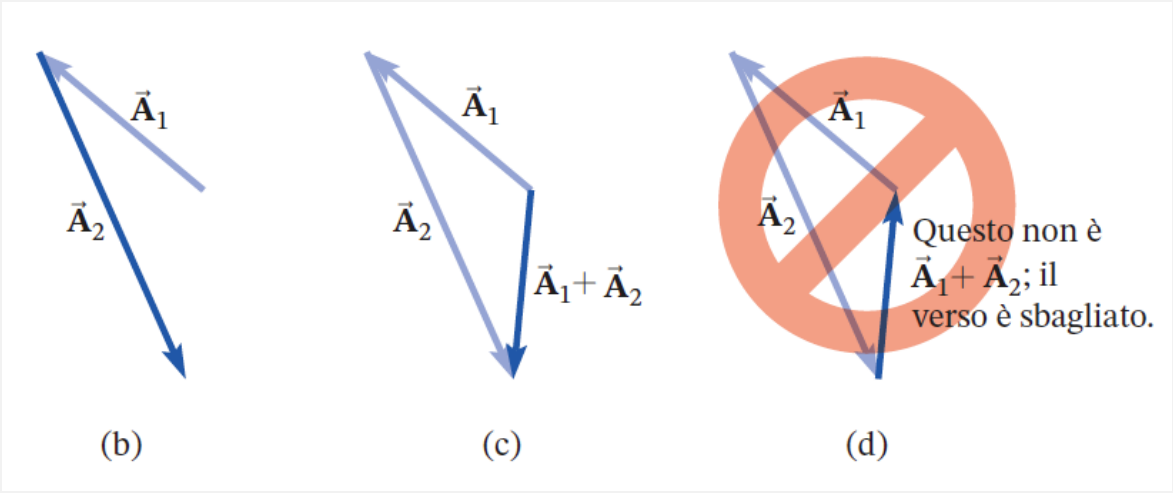


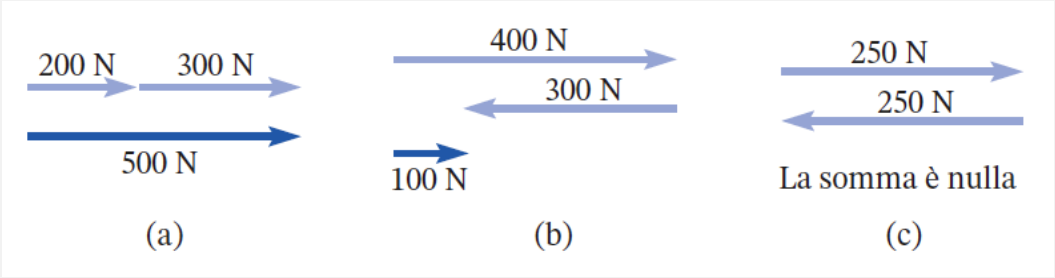
The resultant vector from the addition does not depend on the order of the vectors being added.



VECTOR OPERATIONS

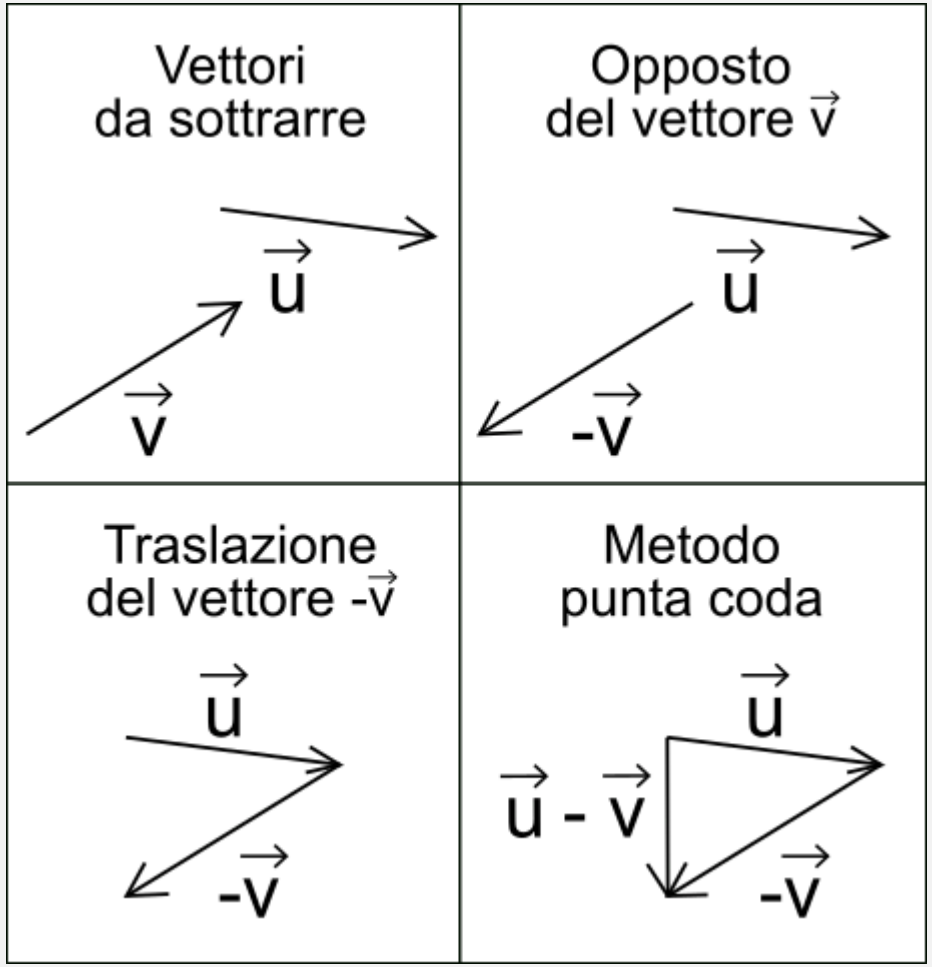
Vector Addition

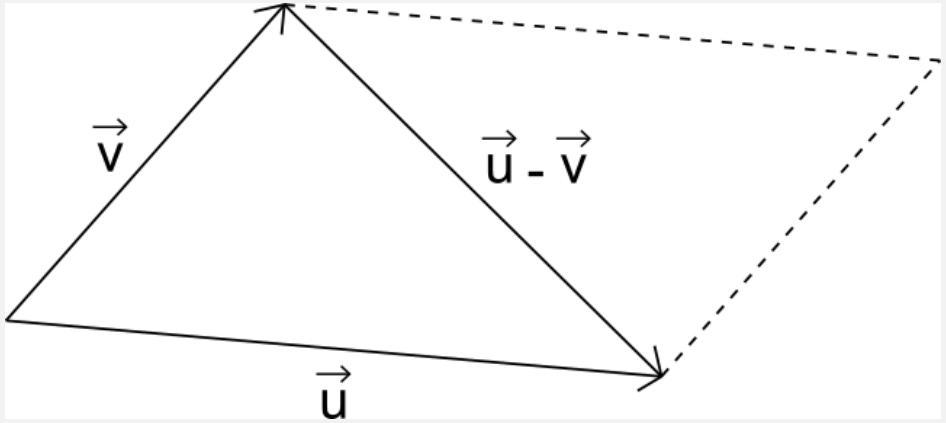


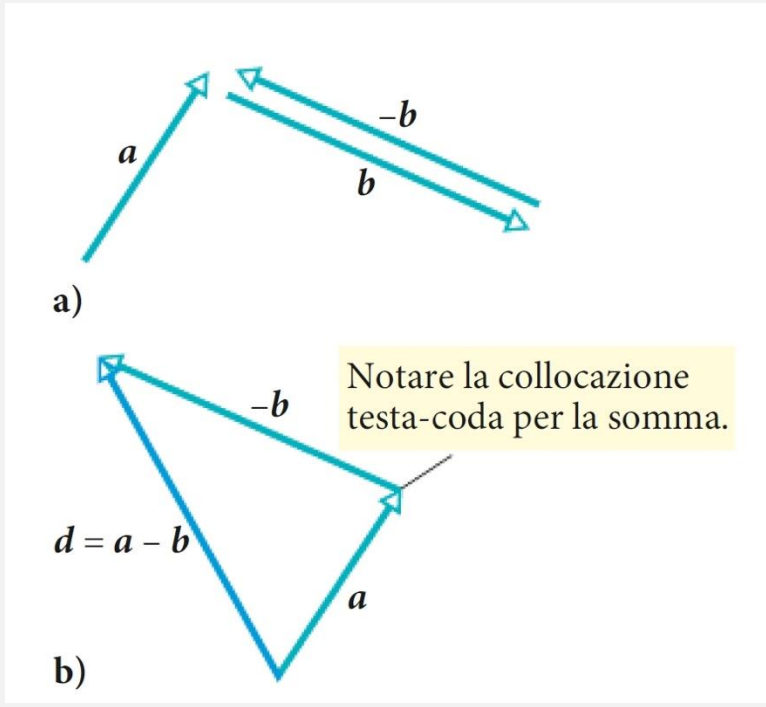


VECTOR OPERATIONS

Vector Subtraction

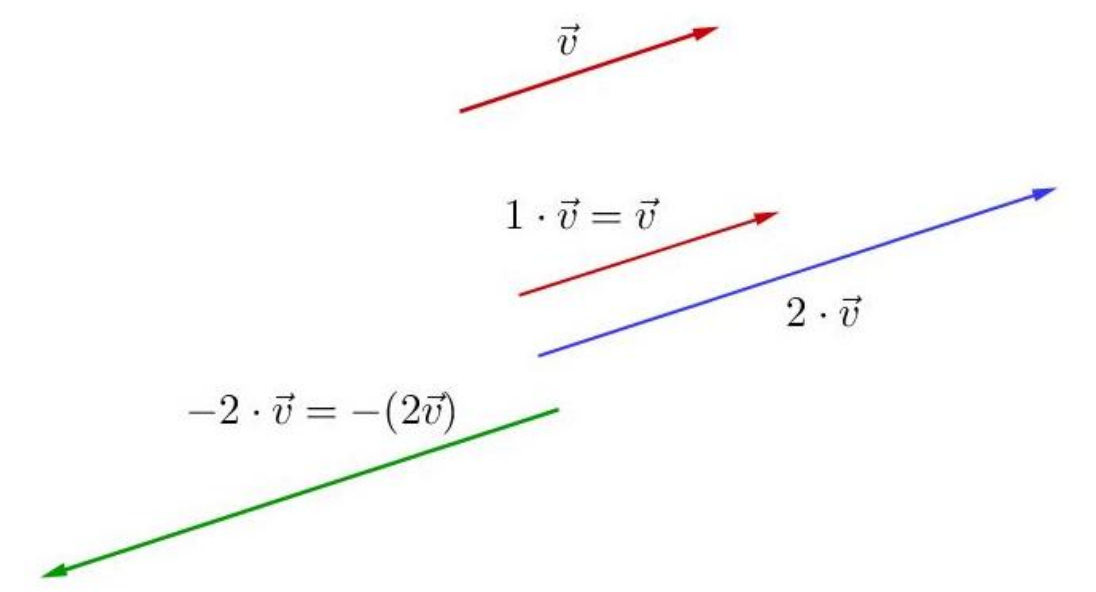






VECTOR OPERATIONS

Vector–Scalar Product



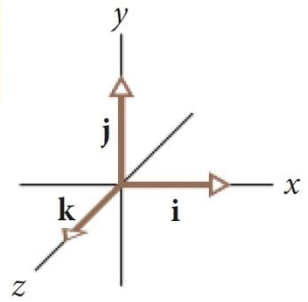
The vector $\vec{v}$ has

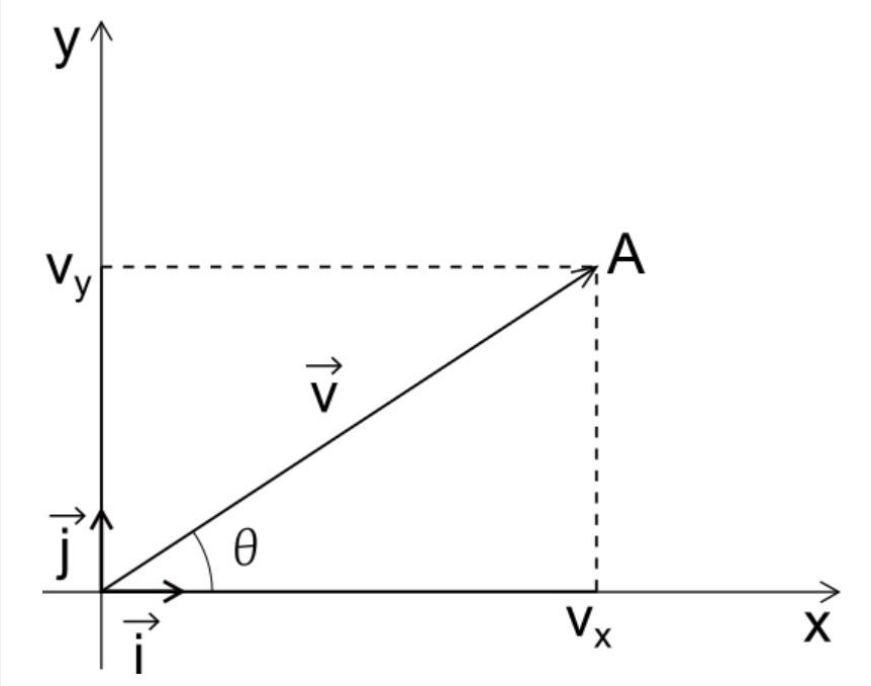
- the same direction as $\vec{u}$
- the same sense as $\vec{u}$ if $k > \mathbf{0}$, opposite sense if $k < \mathbf{0}$
- magnitude $k|\vec{u}| = ku$

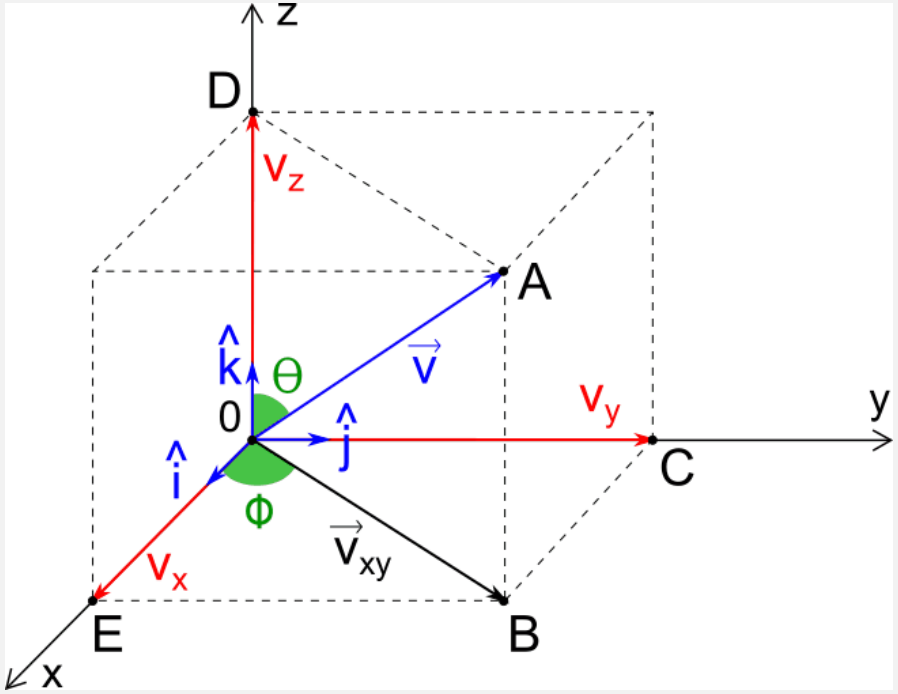
OPERATIONS BETWEEN VECTORS

Vector Components / Projections on x and y

I versori sono diretti come gli assi.

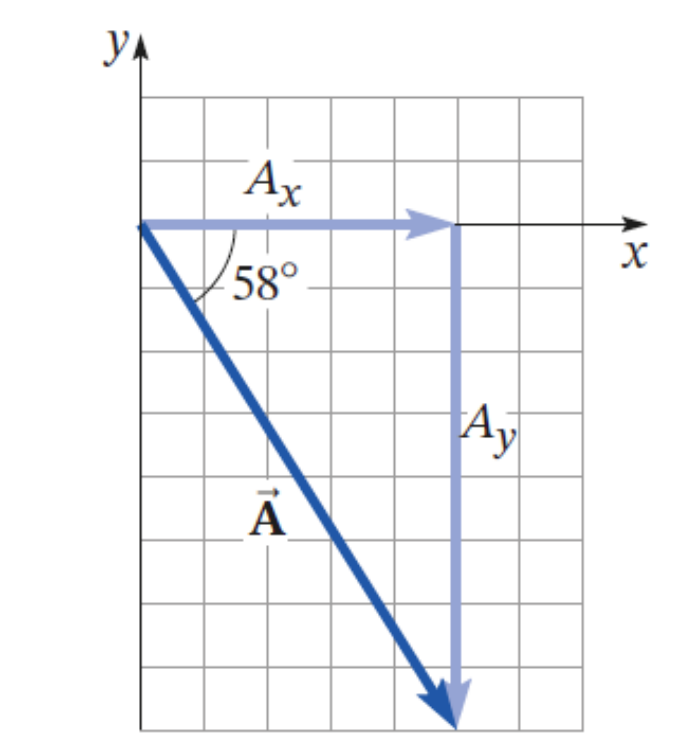






OPERATIONS BETWEEN VECTORS

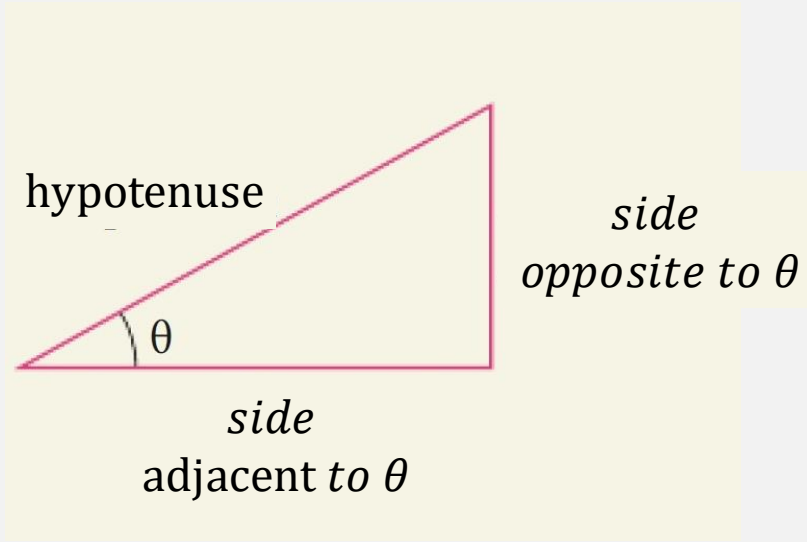
Vector Components / Projections on x and y



$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta}$$



VECTOR OPERATIONS

Algebra with Components

$$\boldsymbol{r} = \boldsymbol{a} + \boldsymbol{b}$$

$$r_x = a_x + b_x$$

$$r_y = a_y + b_y$$

$$r_z = a_z + b_z$$

r is equal to the vector (a+b): if this is true, each component of r must coincide with the corresponding component of (a+b)

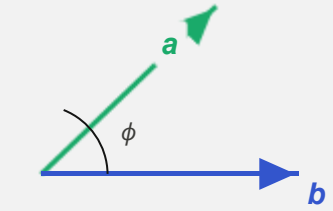
Two vectors are equal if their respective components are all equal to one another

$$\boldsymbol{d} = \boldsymbol{a} + (-\boldsymbol{b}) \longrightarrow d_x = a_x - b_x \qquad d_y = a_y - b_y \qquad d_z = a_z - b_z$$

where $\boldsymbol{d} = d_x\boldsymbol{i} + d_y\boldsymbol{j} + d_z\boldsymbol{k}$

VECTOR OPERATIONS

Dot Product

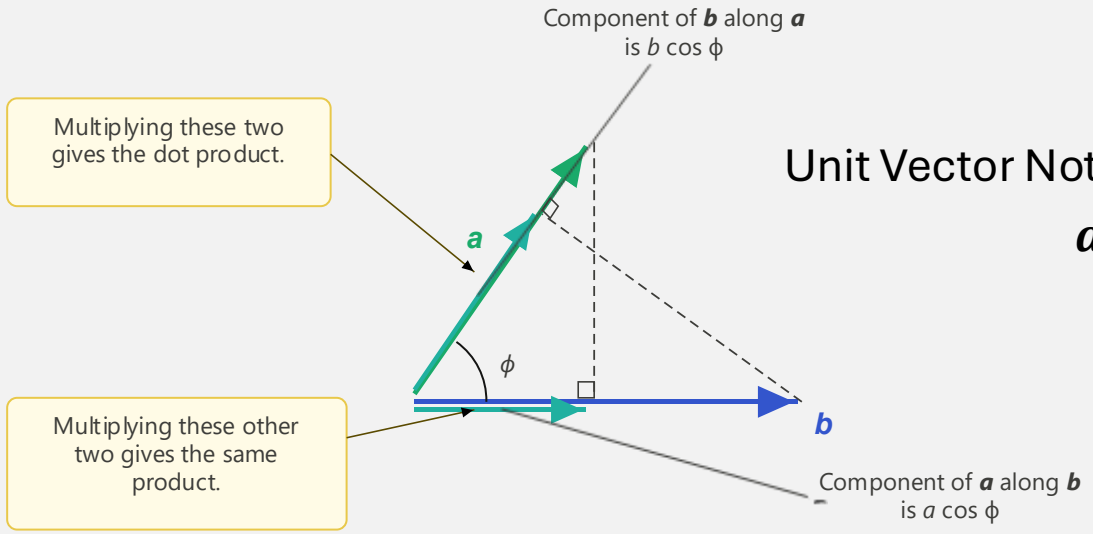


a)

Dot Product of Vectors a and b («a dot b»):

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\,cos\varphi$$

All the terms on the right-hand side are scalars; therefore, the product $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar!



b)

Unit Vector Notation:

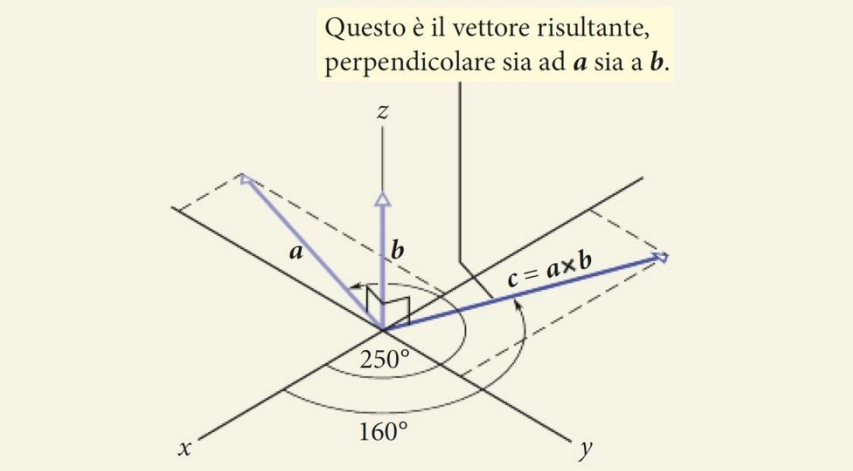
$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \cdot (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k})$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

[Distributive Property]

VECTOR OPERATIONS

Vector Product

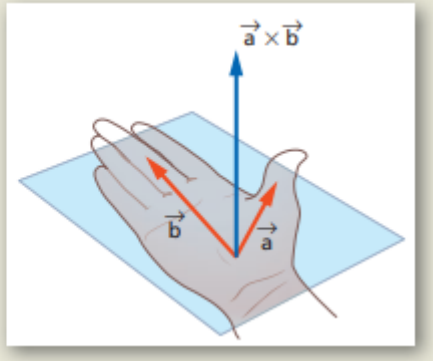


Prodotto vettoriale

Il prodotto vettoriale di due vettori $\vec{a}$ e $\vec{b}$ è il vettore $\vec{c}$ che ha:

- **modulo** uguale ad $ab \operatorname{sen} \alpha$;
- **direzione** perpendicolare al piano individuato dai due vettori;
- **verso** dato dalla regola della mano destra, illustrata nella figura.

Si indica con $\vec{a} \times \vec{b}$.



Vector product of vectors a and b (“a vector b”) → vector c, whose magnitude is:

$$\boldsymbol{c = ab\ sin\varphi}$$

No Commutative Property: $\boldsymbol{a \times b \neq b \times a}$

Yes Distributive Property: $\boldsymbol{a \times (b + c) = a \times b + a \times c}$

With a // b → a × b = 0

With a ⊥ b → |a × b| = max

Example — swimming across a flow (relative velocity)

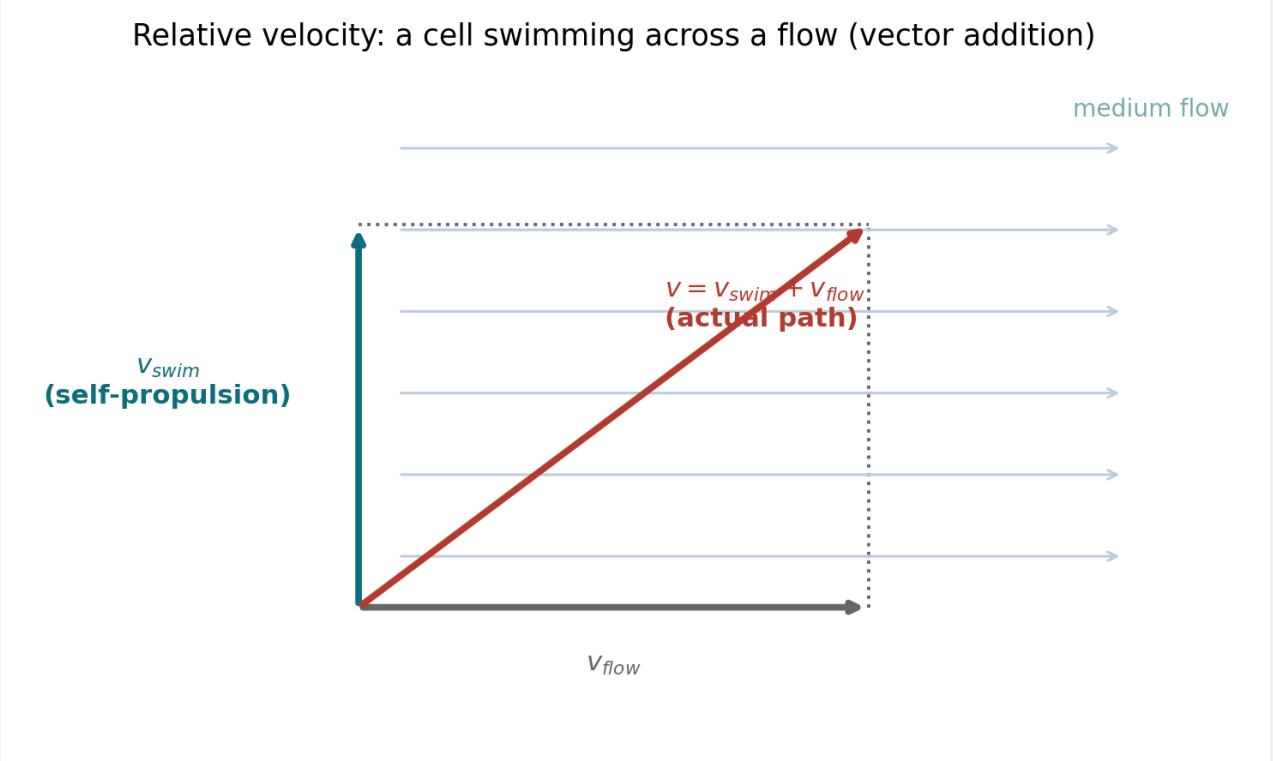
A motile cell or microorganism in a moving medium does not travel where it 'aims'.

Its true velocity is the vector sum of self-propulsion and the medium's flow:

$$v = v_{\mathrm{swim}} + v_{f\mathrm{low}}$$

The same rule governs a particle steered in a microfluidic channel.

Vectors add tip-to-tail; only the resultant tells you the real trajectory.



In practice — motility in a flowing medium

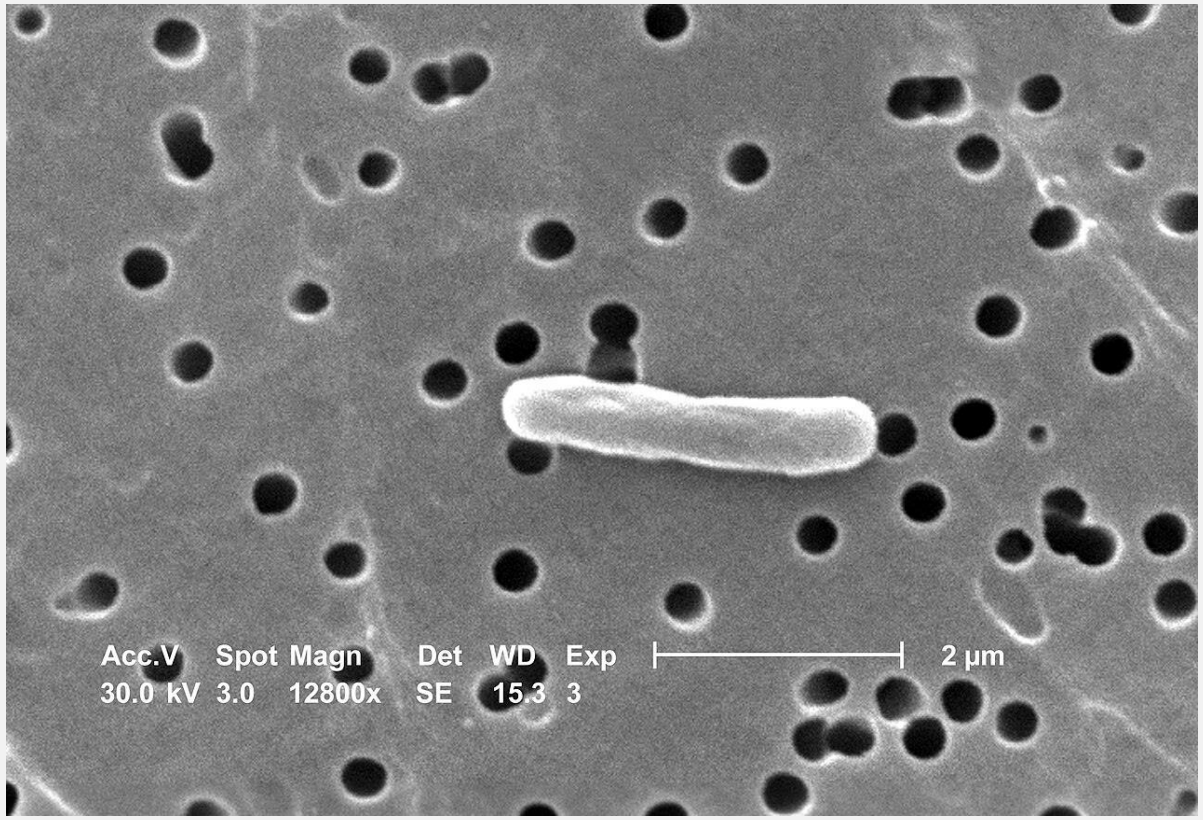
E. coli swims at ~20 µm/s with rotating flagella; sperm cells reach ~50 µm/s.

If the medium flows at 40 µm/s while a cell swims across at 30 µm/s:

resultant = $\sqrt{(30^2 + 40^2)} = 50$ µm/s, at $\arctan(30/40) \approx 37°$

The cell is carried downstream — to go straight it must 'aim upstream'.

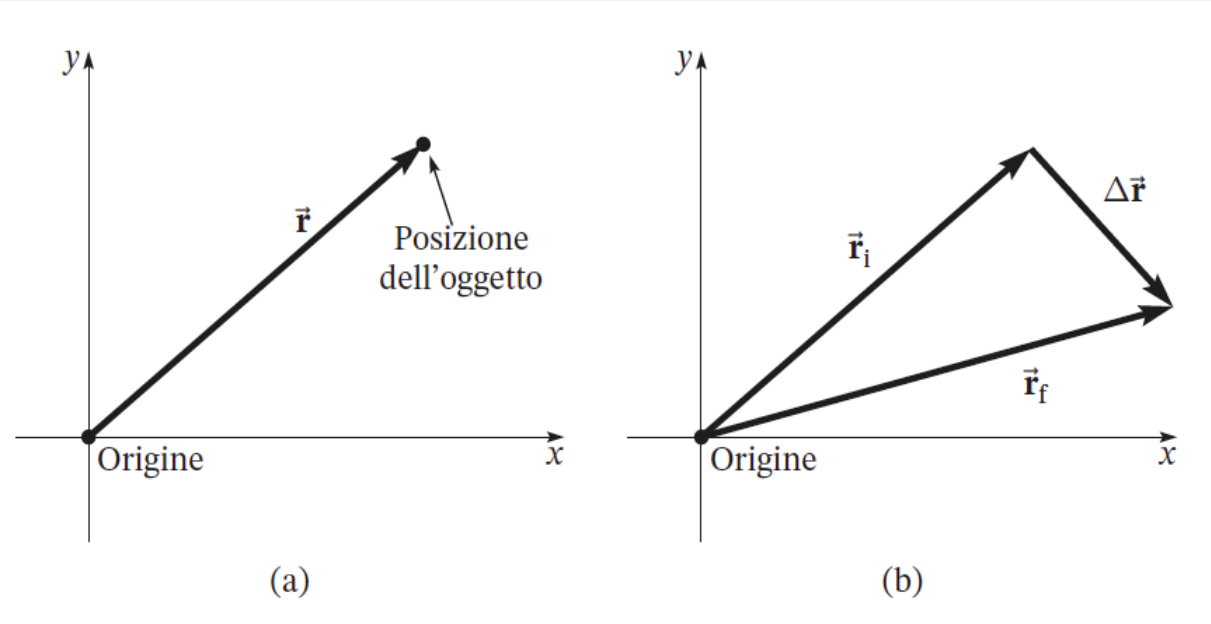
Microfluidic devices exploit this to steer and separate cells in flow.

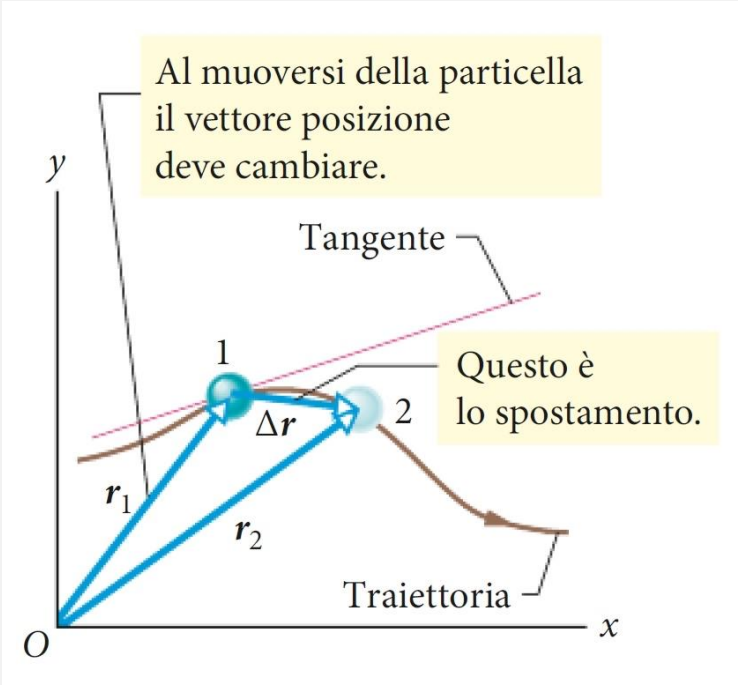


E. coli (SEM) — CDC, public domain

POSITION, DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

Position Vector – Displacement Vector





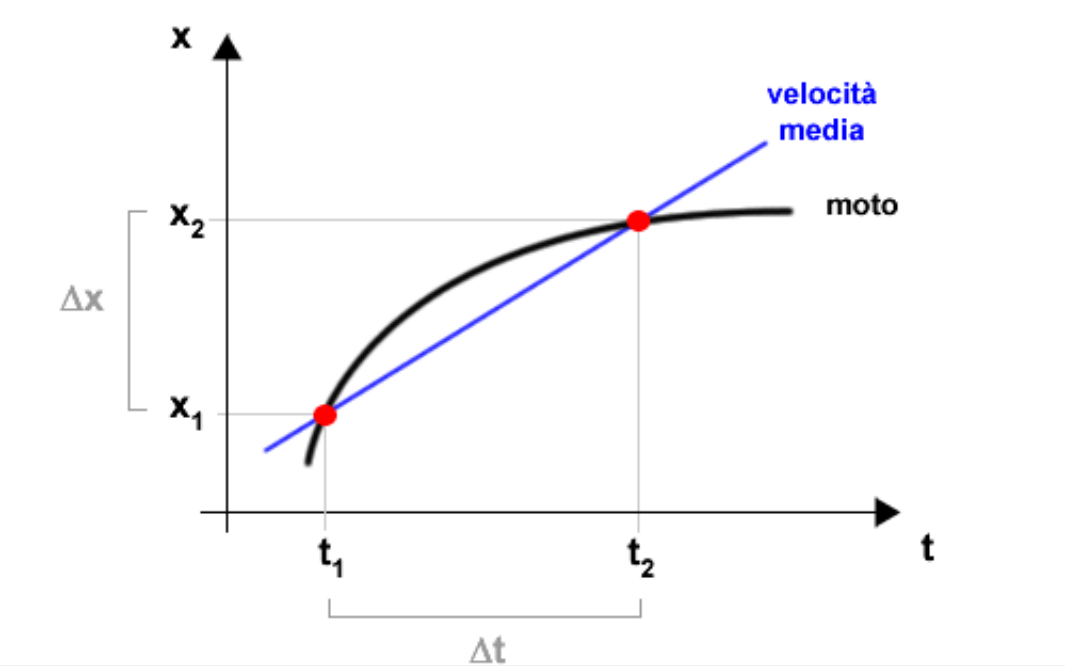
$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$ Position Vector

$$\Delta\vec{r} = (x_f - x_i)\cdot\hat{\imath} + (y_f - y_i)\cdot\hat{\jmath} = \Delta x\cdot\hat{\imath} + \Delta y\cdot\hat{\jmath}$$

Displacement Vector

POSITION, DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

Average Velocity Vector



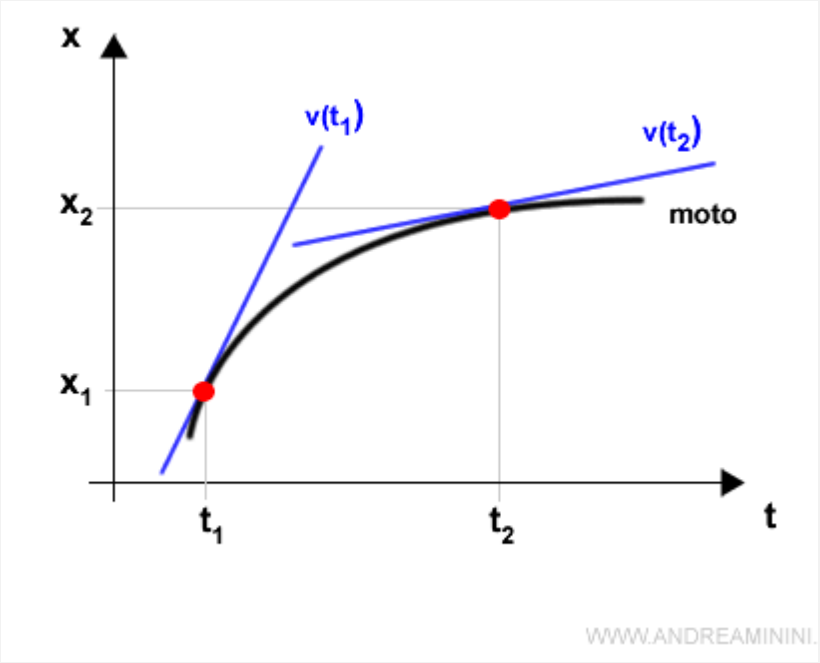
$$\vec{v}_m = \frac{\Delta\vec{r}}{\Delta \mathrm{t}} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} \qquad [L]/[t] = [Lt^{-1}]$$

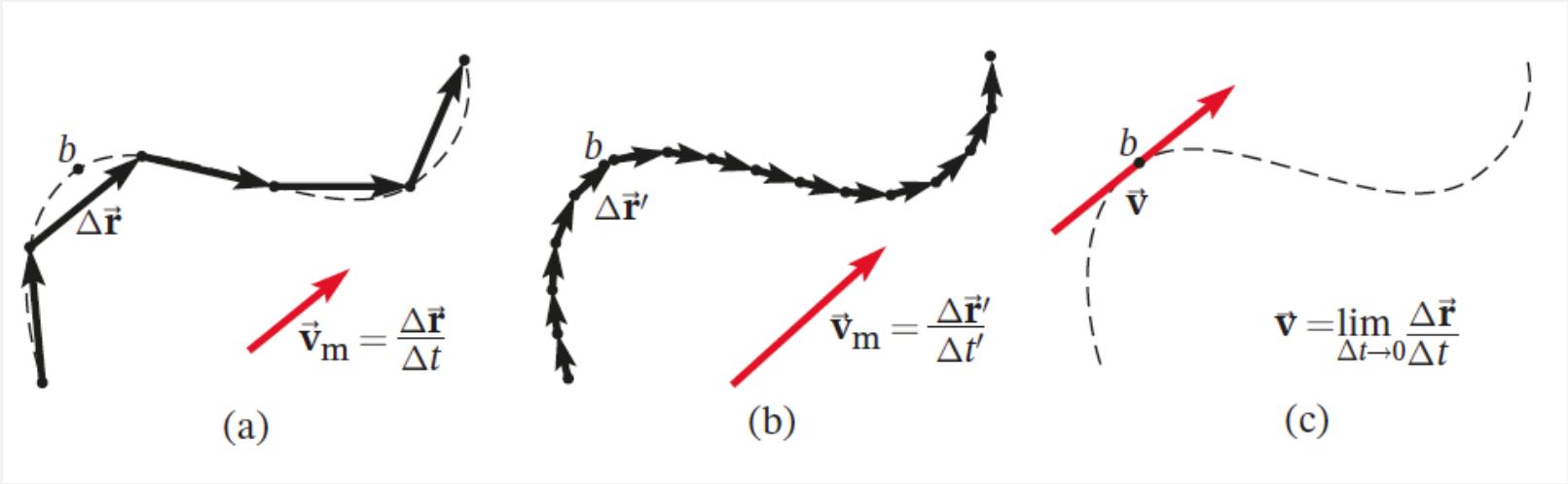
$$\vec{v}_m = \frac{\Delta x\boldsymbol{i} + \Delta y\boldsymbol{j} + \Delta z\boldsymbol{k}}{\Delta \mathrm{t}} = \frac{\Delta x}{\Delta t}\boldsymbol{i} + \frac{\Delta y}{\Delta t}\boldsymbol{j} + \frac{\Delta z}{\Delta t}\boldsymbol{k}$$

$$v_{m,x} = \frac{\Delta x}{\Delta t} \ , \qquad v_{m,y} = \frac{\Delta y}{\Delta t}$$

POSITION, DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

Instantaneous Velocity Vector





In Scalar Form:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

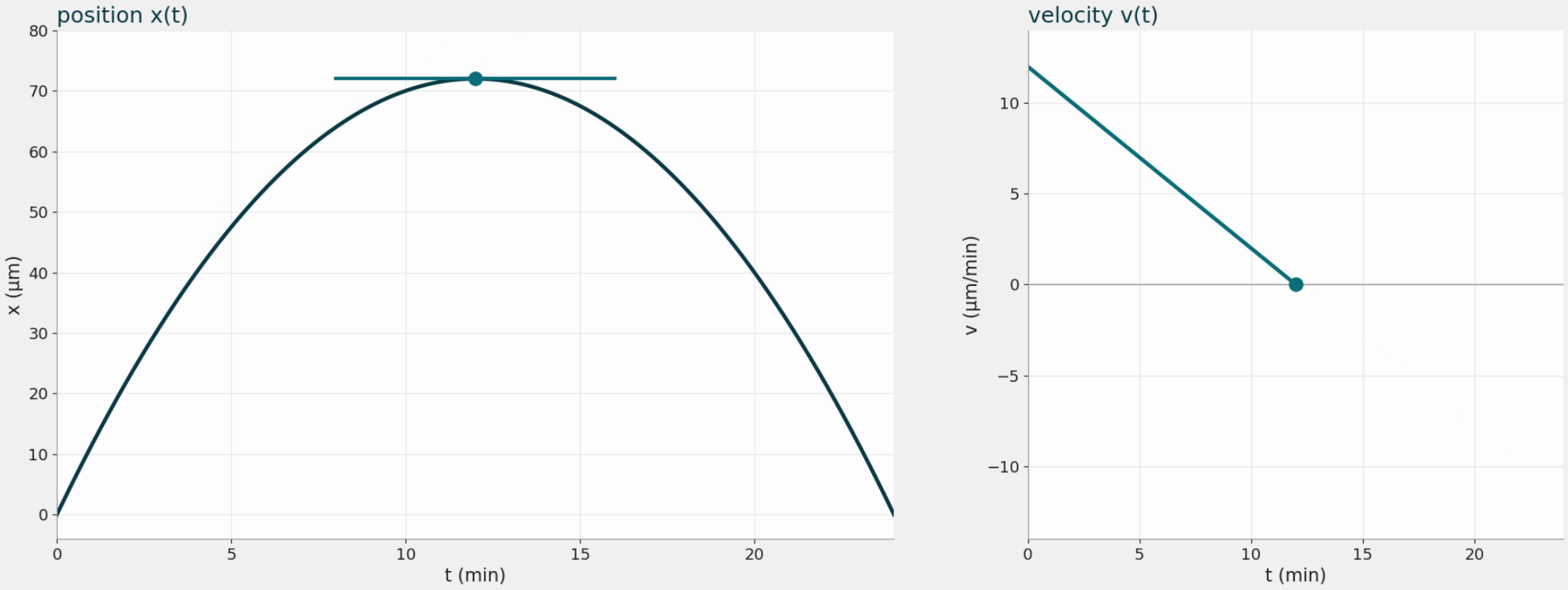
$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \quad , \qquad v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

Same direction and same sense as the displacement at that instant

Animation — average and instantaneous velocity

Average and instantaneous velocity

A cell moves along a straight line: $x(t) = 12t - 0.5t^2$ (x in µm, t in min)



t = 12.0 min x = 72.0 µm

v = dx/dt = 12 − t = 0.0 µm/min

x is at its maximum → horizontal tangent → v = 0: the cell reverses its motion

Example — cell migration: displacement vs path

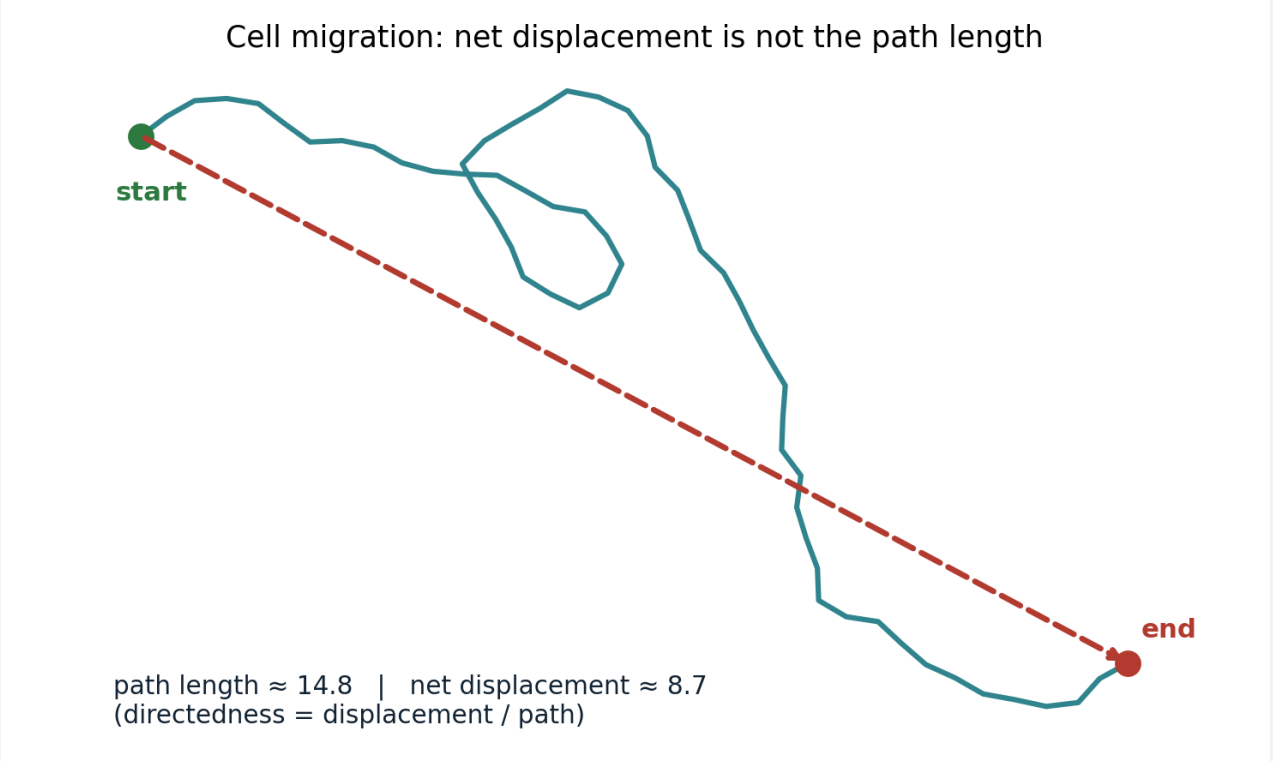
In a chemotaxis or wound-healing assay, cells wander but tend to drift in one direction.

The net displacement (start → end) is much shorter than the total path length.

Average velocity = net displacement / time — a vector, not path length / time.

Their ratio, the 'directedness', measures how purposeful the migration is.

Cell-tracking software computes exactly these vectors from time-lapse microscopy.



In practice — quantifying cell migration

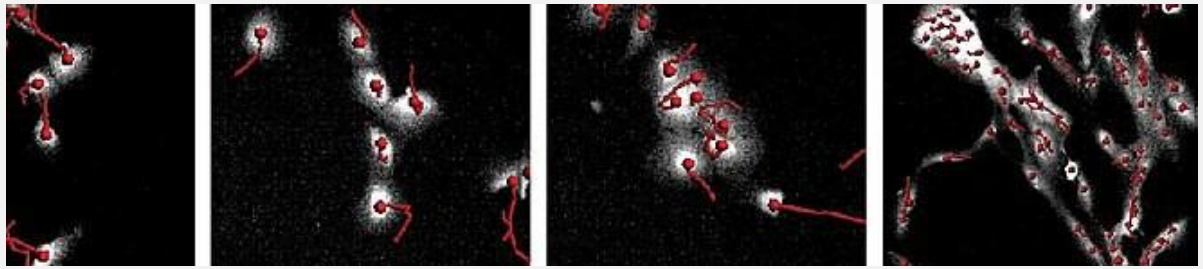
Time-lapse microscopy records each cell's position frame by frame.

If a cell's path totals 150 µm but start and end are 60 µm apart:

directedness = displacement / path = 60 / 150 = 0.40

Average velocity uses displacement (a vector); average speed uses path length.

These metrics quantify chemotaxis, wound healing and cancer-cell invasion.



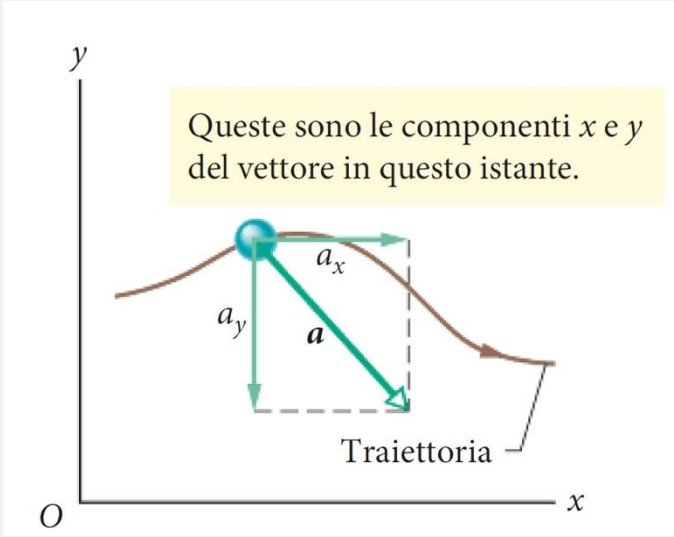
Cell migration tracks — Leggett et al., CC BY 4.0

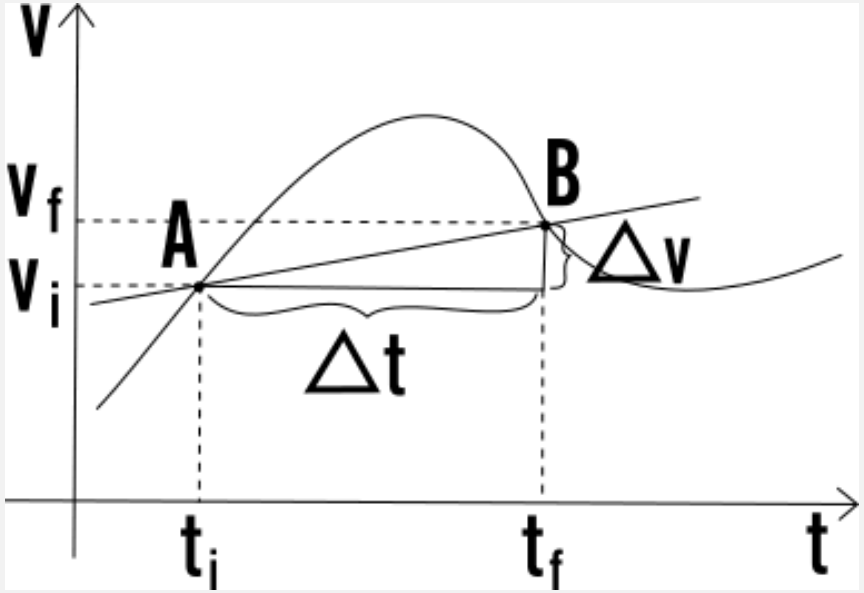
POSITION, DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

Average Acceleration Vector

$$\vec{a}_m = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\Delta\vec{v}}{\Delta t} \qquad a_{m,x} = \frac{\Delta v_x}{\Delta t}\ , a_{m,y} = \frac{\Delta v_y}{\Delta t}$$

Same direction and same sense as $\Delta\vec{v}[Lt^{-1}]/[t] = [Lt^{-2}]$



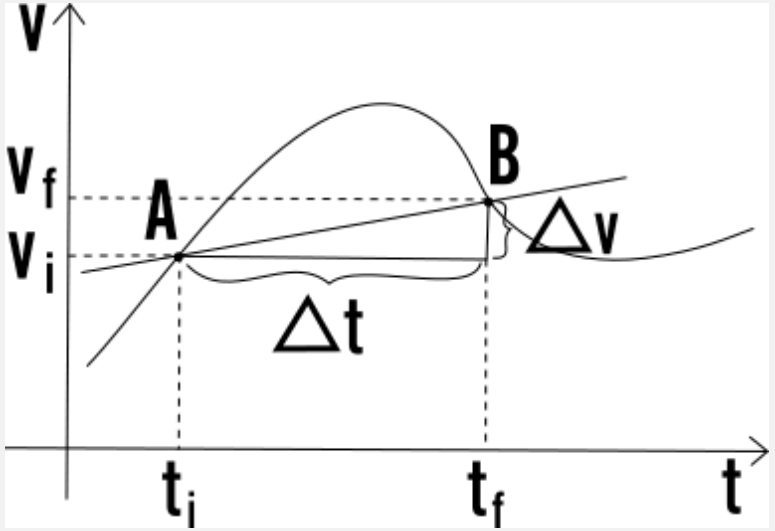


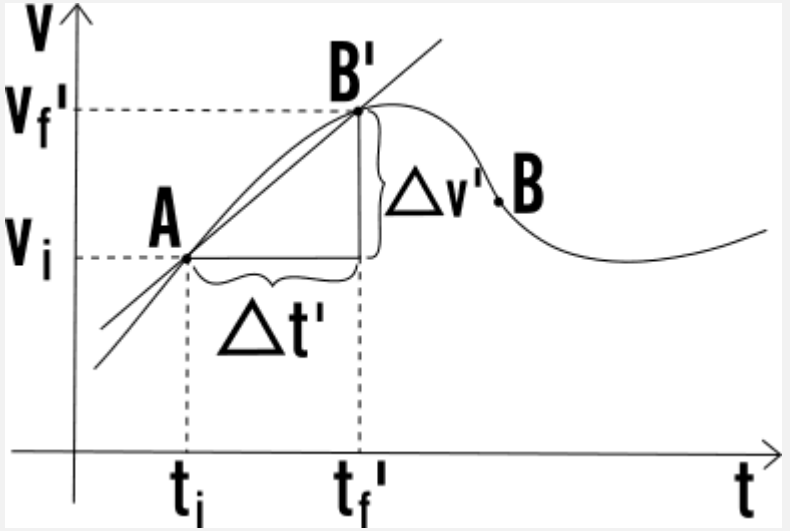
POSITION, DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

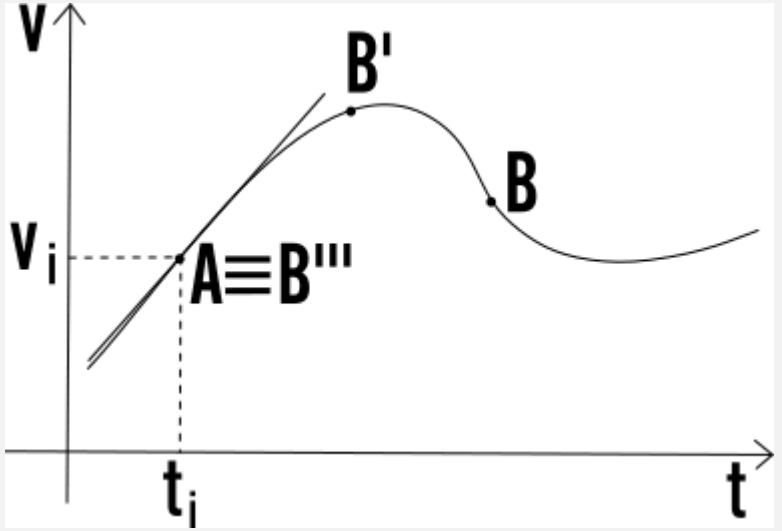
Instantaneous Acceleration Vector

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} \quad , \qquad a_y = \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}$$







Velocity and Acceleration as Derivatives

If position is known as a function of time, velocity and acceleration are its derivatives (module M5):

$v(t) = dx/dt$ — slope of the tangent to x(t)

$a(t) = dv/dt = d^2x/dt^2$ — slope of the tangent to v(t)

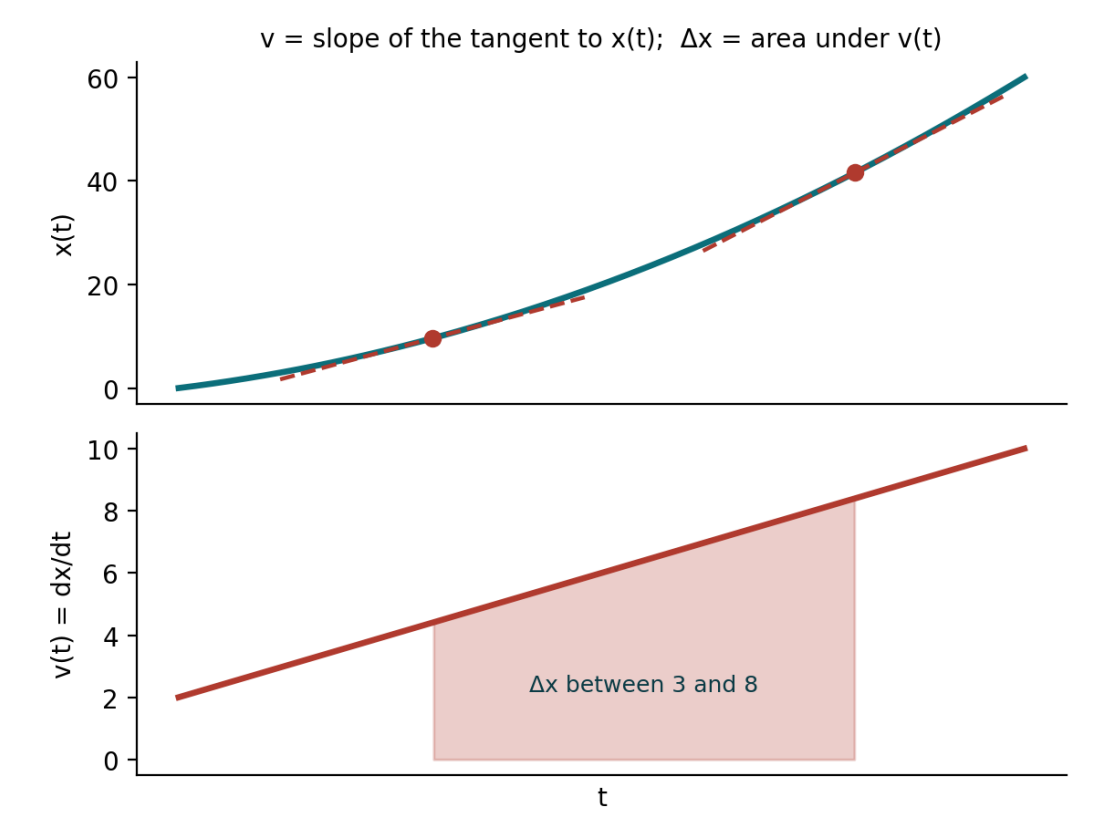
The inverse path is the integral (module M6):

$\Delta x = \int v\, dt$ — signed area under v(t)

$\Delta v = \int a\, dt$ — signed area under a(t)

In 2D we differentiate component by component: $v_x = dx/dt$, $v_y = dy/dt$.

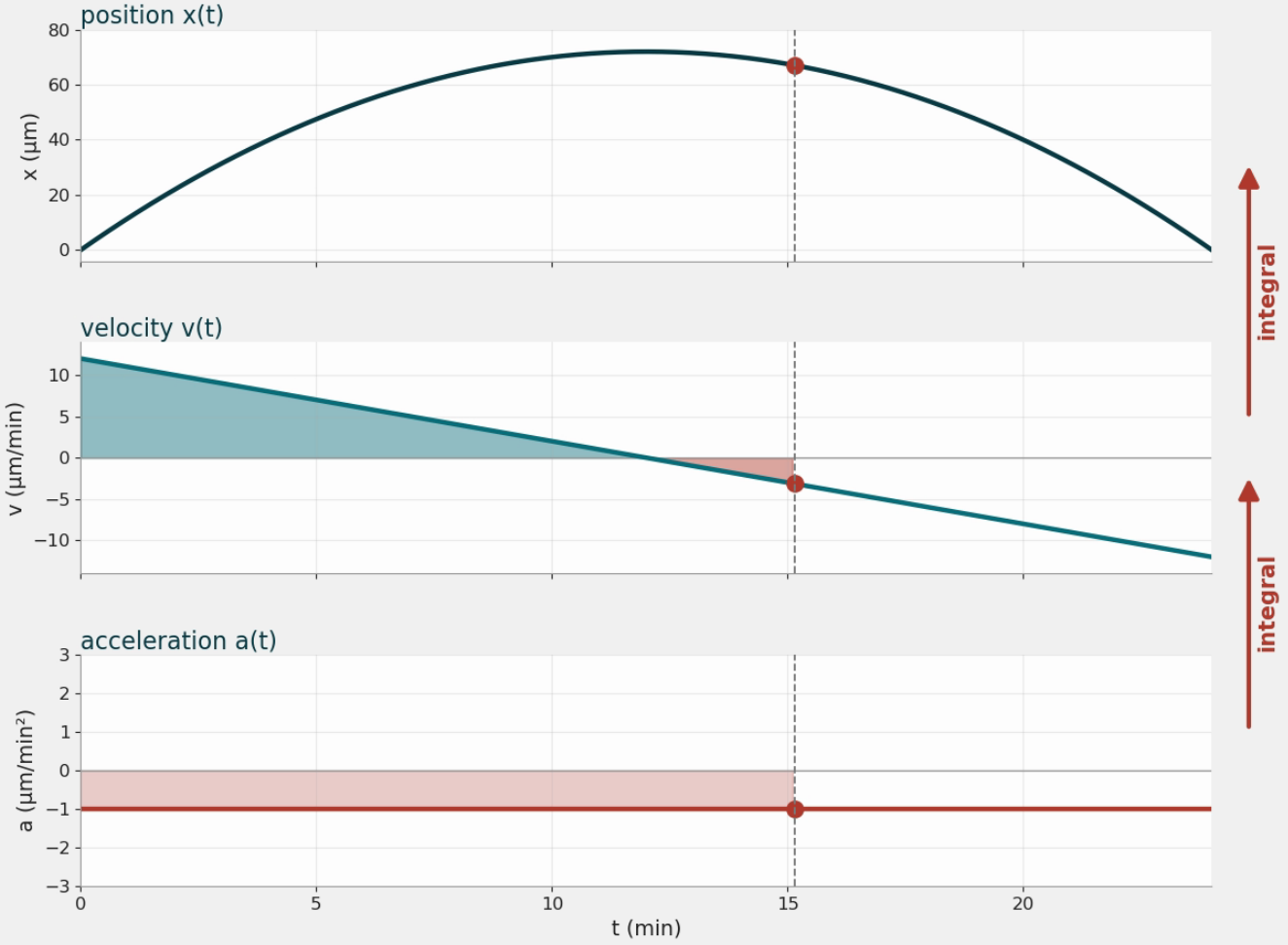
With constant a we get $v = v_0 + at$ and $x = x_0 + v_0t + \frac{1}{2}at^2$: the equations of PHYS3 are derived this way, by integration.



Animation — derivative and integral

Position, velocity, acceleration: derivative and integral

A cell along a straight line: $x(t) = 12t - 0.5t^2$, $v(t) = 12 - t$, $a(t) = -1$ µm/min²



Integral: the area

area under a(t) → Δv
Δv = −15.1 µm/min

signed area under v(t) → Δx

$\int_0^t v\,dt = 67.1$ µm = x(t) − x(0)

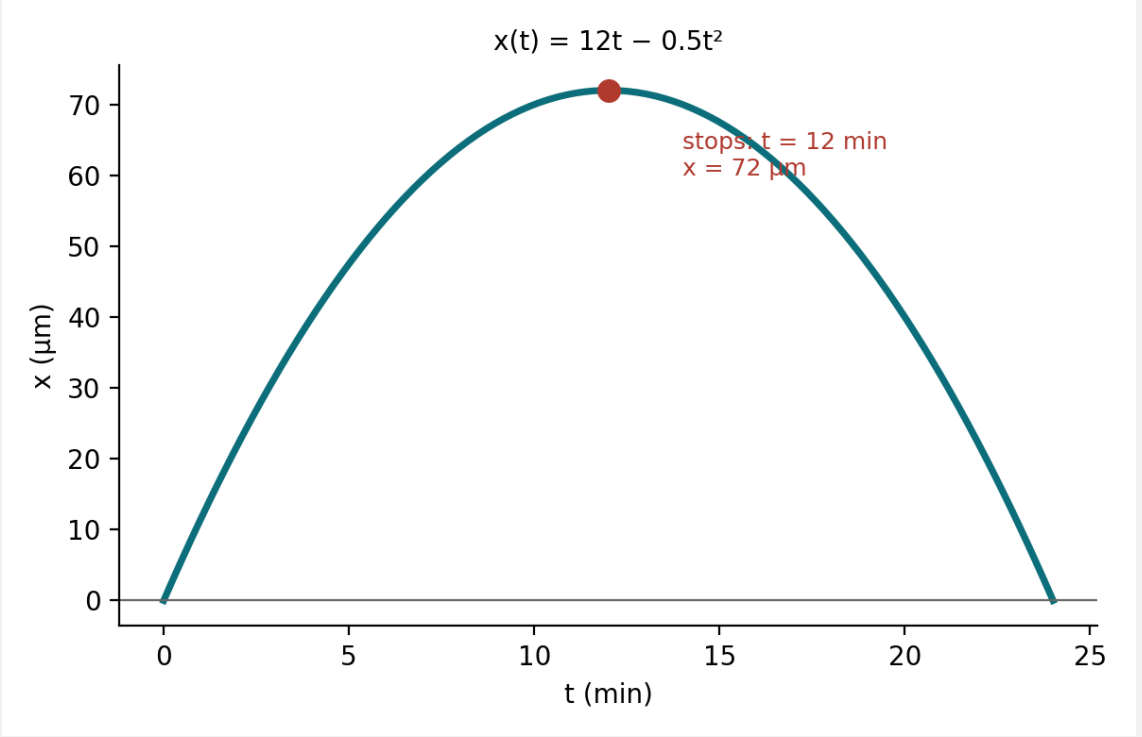
path length $\int |v|\,dt$ = 76.9 µm

Red area: backward motion,
negative in ∫v dt,
positive in ∫|v| dt

Advanced exercise — derivative and integral

A cell moves along the x axis with $x(t) = 12t - 0.5t^2$ (x in µm, t in min), for $0 \le t \le 24$ min.

Find: (a) v(t) and a(t); (b) v at t = 4 min; (c) the instant at which the cell stops and its position; (d) the displacement between 0 and 24 min using $\Delta x = \int v\,dt$, and the total path length; (e) what does the result of (d) mean?



Advanced exercise — solution (a–c)

Step 1 — velocity as a derivative (a). $v = dx/dt$. Differentiate term by term: $d(12t)/dt = 12$ and $d(0.5t^2)/dt = 0.5 \cdot 2t = t$. So

$$x(t) = 12t - 0.5t^2 \;\rightarrow\; \mathbf{v(t) = 12 - t} \quad (\mu m/min)$$

Step 2 — acceleration (a). $a = dv/dt$: $d(12)/dt = 0$ and $d(-t)/dt = -1$, so **$a = -1\ \mu m/min^2$**. It is constant and negative: the velocity decreases by 1 µm/min every minute.

Step 3 — velocity at t = 4 min (b). Substitute $t = 4$ in $v(t)$: $v(4) = 12 - 4 =$ **8 µm/min** (positive: the cell is moving forward).

Step 4 — when it stops (c). Set $v = 0$: $12 - t = 0 \;\rightarrow\; t =$ **12 min**. Compute the position at that instant:

$$x(12) = 12 \cdot 12 - 0.5 \cdot 12^2 = 144 - 72 = \mathbf{72\ \mu m}$$

Before 12 min $v > 0$ (the cell moves forward), after $v < 0$ (it moves backward): at $t = 12$ min it reverses its motion.

Advanced exercise — solution (d–e)

Step 5 — net displacement (d). $\Delta x = \int_0^{24} v\,dt$. An antiderivative of $12 - t$ is $F(t) = 12t - t^2/2$ (check by differentiating: $F' = 12 - t$). Compute $F(24) - F(0)$:

$F(24) = 12 \cdot 24 - 24^2/2 = 288 - 288 = 0$; $F(0) = 0$ → **$\Delta x = 0$ µm**

Check: $\Delta x = x(24) - x(0) = (12 \cdot 24 - 0.5 \cdot 24^2) - 0 = 0$.

Step 6 — path length (d). Since v changes sign at 12 min, split the two stretches (otherwise the areas cancel):

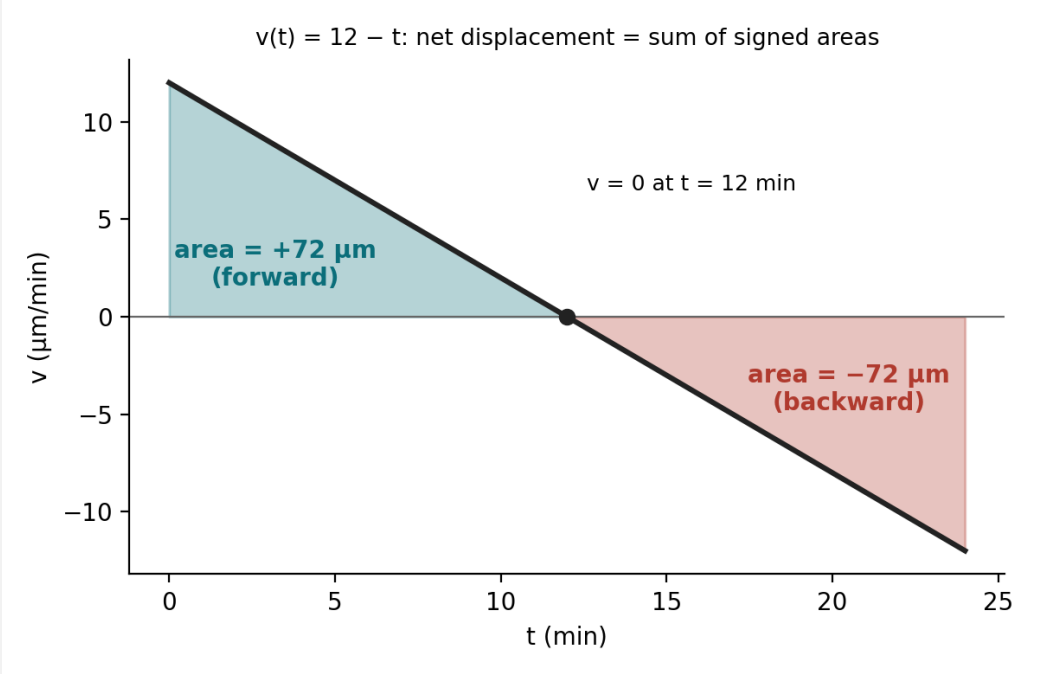
0–12 min: $x(12) - x(0) = +72$ µm (forward)

12–24 min: $x(24) - x(12) = 0 - 72 = -72$ µm (backward)

path = $|72| + |-72|$ = **144 µm**

(equal to the area of the two triangles of v(t): ½ · 12 min · 12 µm/min = 72 µm each).

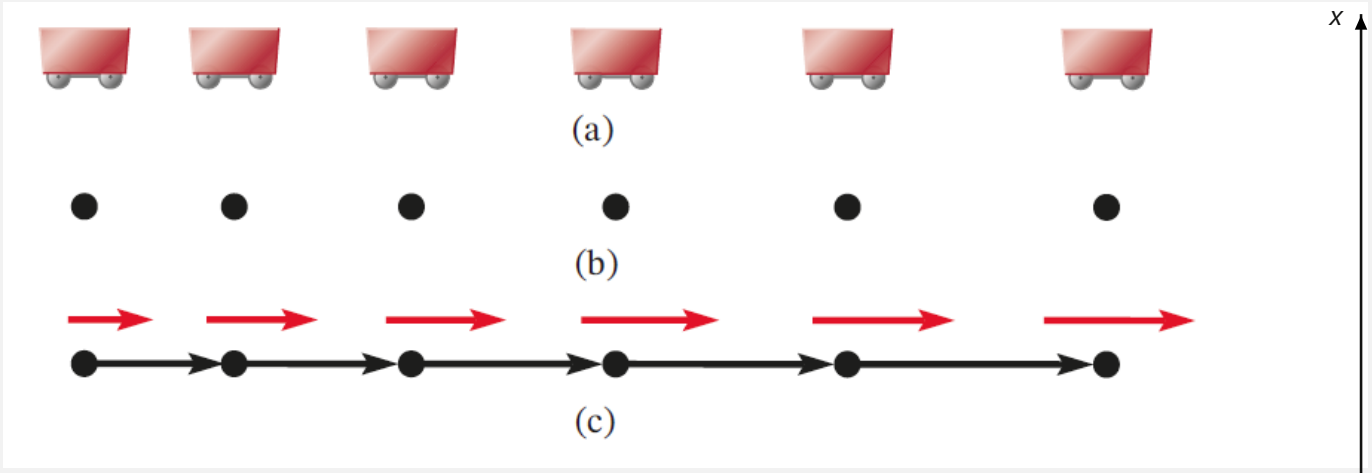
Step 7 — meaning (e). The cell returns to its starting point: zero net displacement and zero vector average velocity $\Delta x/\Delta t$. The average speed is path/time = 144/24 = **6 µm/min**.

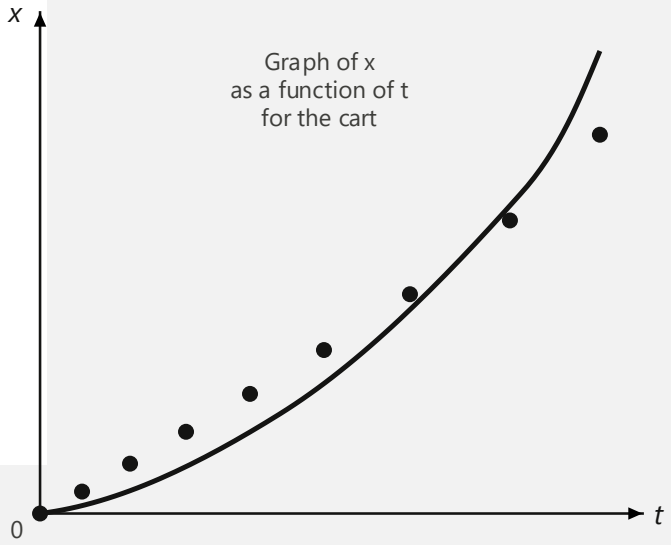


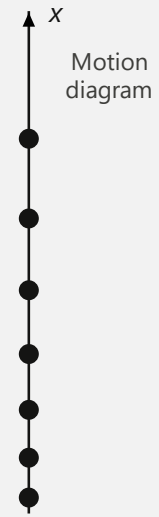
POSITION, DISPLACEMENT, VELOCITY AND ACCELERATION VECTORS

Motion Diagram

A motion diagram shows an object's position as a function of time

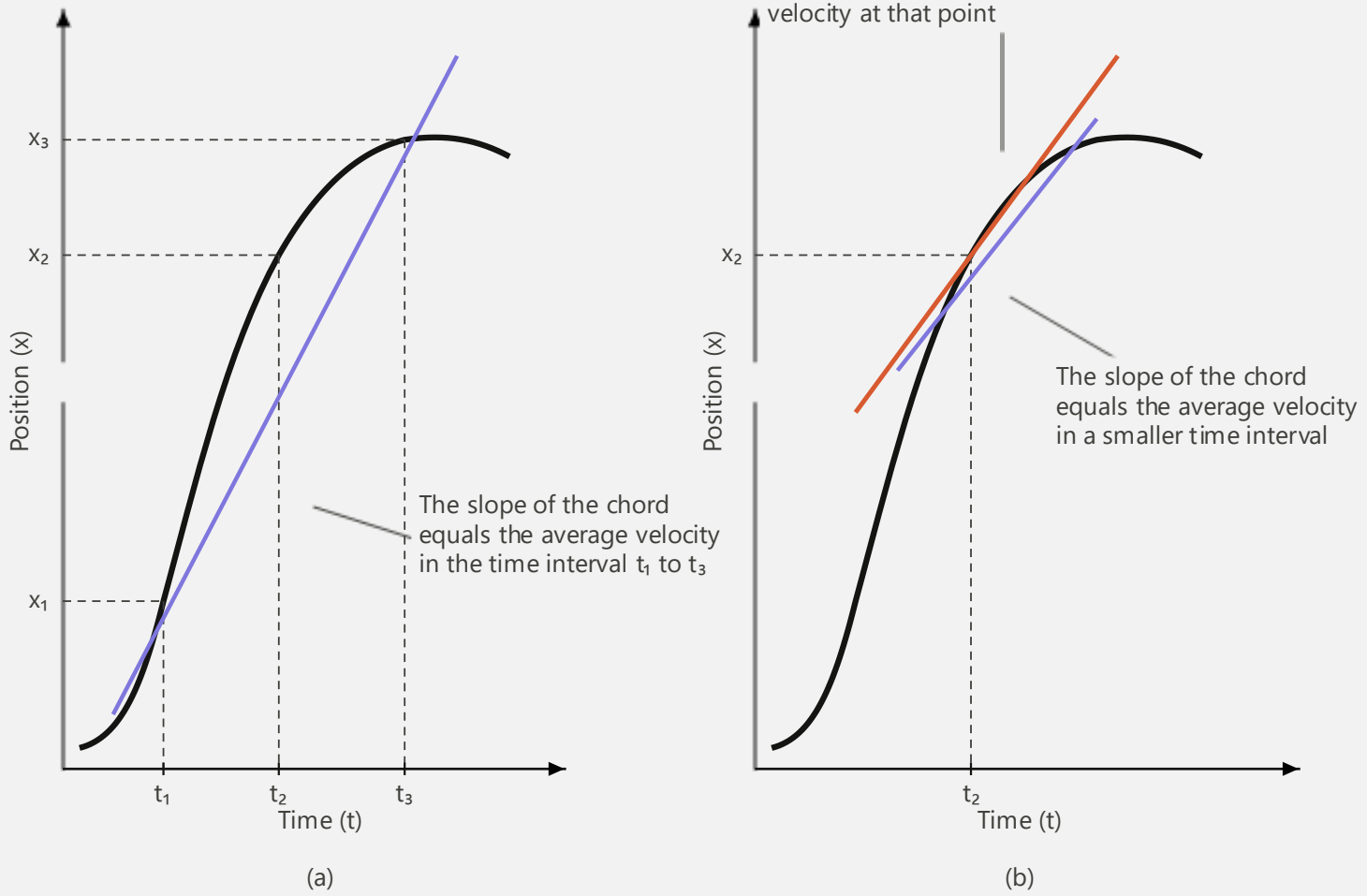






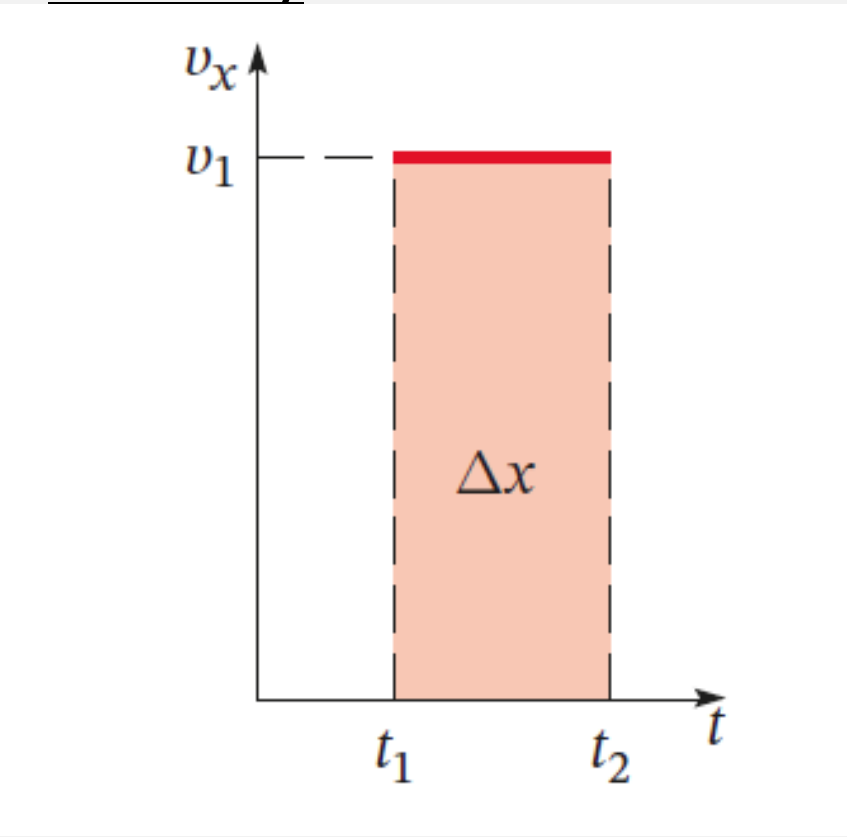
POSITION, DISPLACEMENT, VELOCITY AND ACCELERATION VECTORS

Graphical Relationship Between Position and Velocity



POSITION, DISPLACEMENT, VELOCITY, AND ACCELERATION VECTORS

Graphical Relationship Between Position, Time, and Velocity

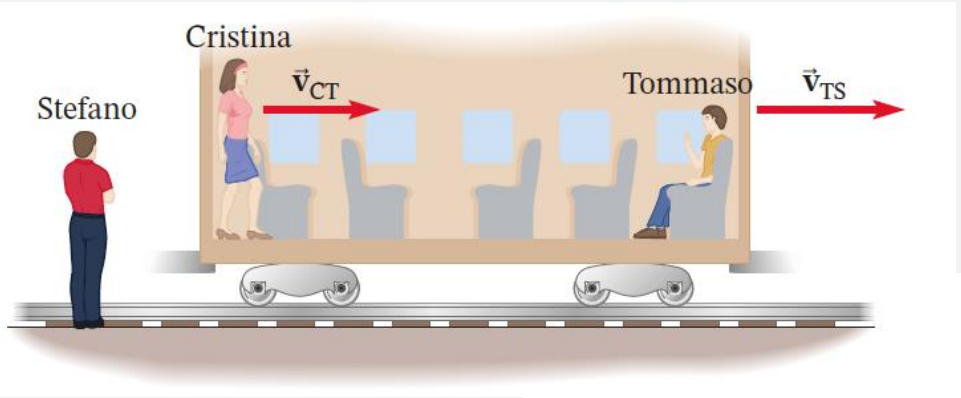


$$v_x = v_{m,x} = \frac{\Delta x}{\Delta t}$$

$$\Delta x = v_x \Delta t \text{ (for constant } v_x)$$

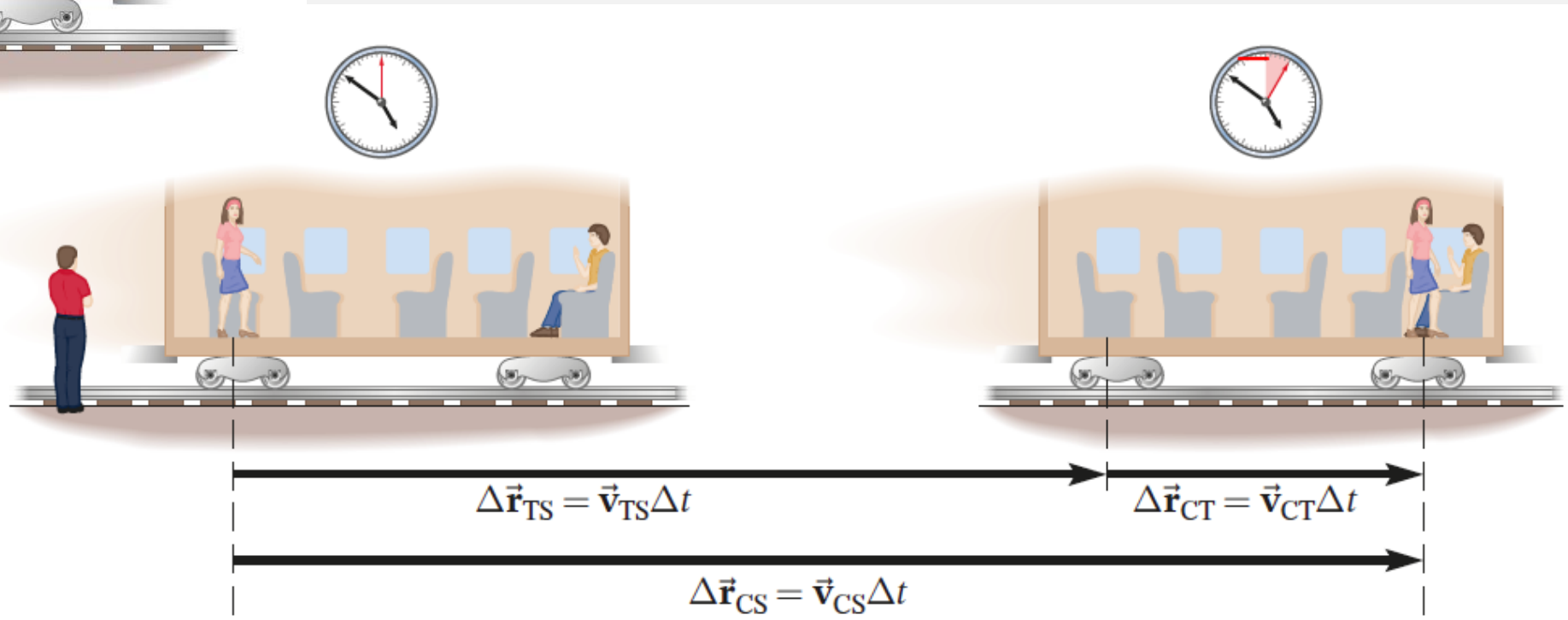
POSITION AND DISPLACEMENT VECTORS, VELOCITY AND ACCELERATION

Relative Velocity and Reference Frames



$$\Delta\vec{r}_{CS} = \Delta\vec{r}_{CT} + \Delta\vec{r}_{TS}$$

$$\vec{v}_{CS} = \vec{v}_{CT} + \vec{v}_{TS}$$

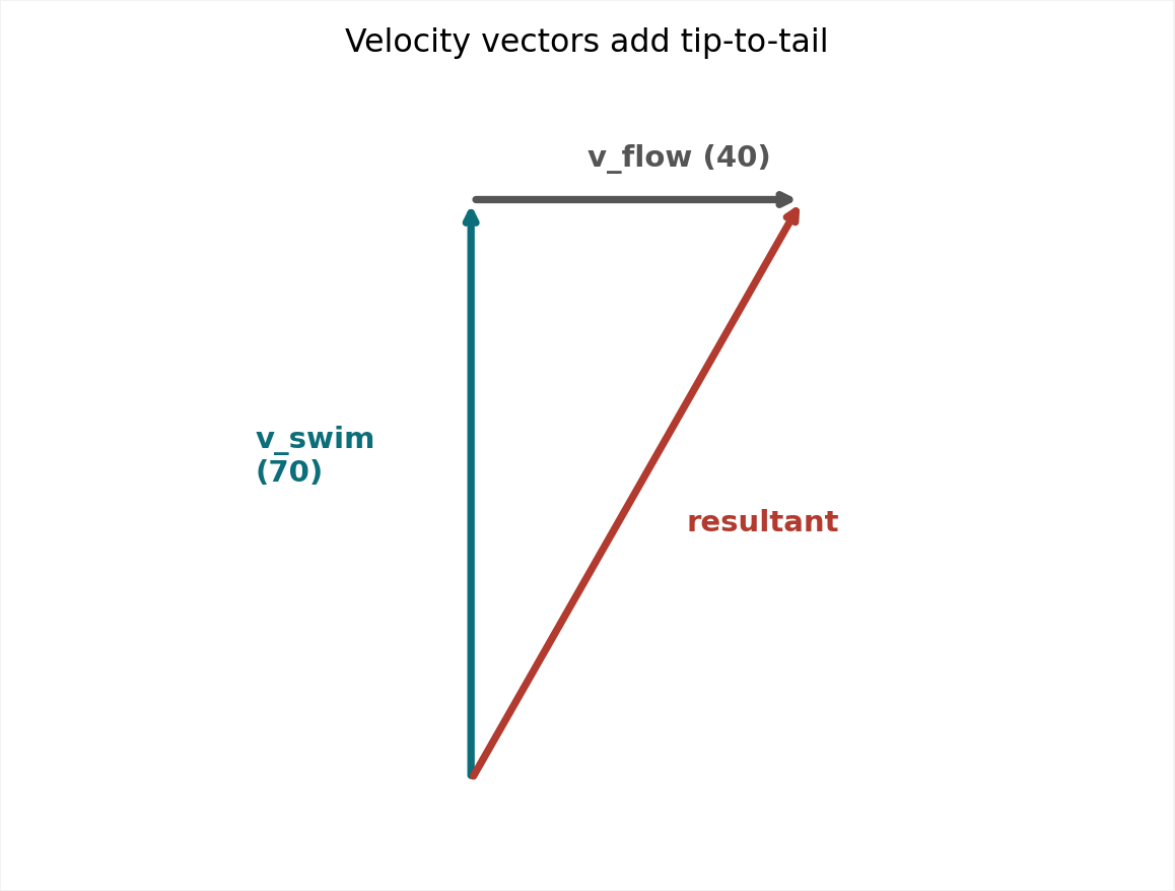


Exercise 1 — swimming across a flow (relative velocity)

A cell swims at 70 µm/s; the medium flows across its path at 40 µm/s.

Find (a) the resultant speed and angle if it aims straight across;

(b) the angle it must aim upstream to move straight across; (c) its net crossing speed then.



Exercise 1 — solution (a)

Step 1 — set up the problem. The velocity of the cell relative to the lab is the vector sum of its swimming velocity and the flow velocity (relative velocity): $v_{res} = v_{swim} + v_{flow}$. Choose two axes: x along the flow, y across it. In case (a) the cell points straight across, so the two vectors are perpendicular:

v_{swim} = 70 µm/s (along y) v_{flow} = 40 µm/s (along x)

Step 2 — magnitude of the resultant. Two perpendicular vectors add with the Pythagorean theorem:

$v_{res} = \sqrt{(70^2 + 40^2)} = \sqrt{(4900 + 1600)} = \sqrt{6500} =$ **80.6 µm/s**

Step 3 — direction. The angle α from the across direction follows from the ratio of the components:

$\tan \alpha = 40 / 70 = 0.571 \rightarrow \alpha = \arctan(0.571) =$ **29.7°** (downstream)

Meaning: the cell reaches the opposite bank displaced downstream; the flow carries it along while it crosses.

Exercise 1 — solution (b–c)

Step 4 — condition for straight-line motion (b). To cross without drifting, the velocity component along x must vanish: the cell must aim slightly upstream, by an angle φ from the across direction, so that its upstream component cancels the flow:

$$v_{swim} \cdot \sin \varphi = v_{flow} \quad \rightarrow \quad 70 \cdot \sin \varphi = 40$$

Step 5 — solve for φ. $\sin \varphi = 40 / 70 = 0.571 \quad \rightarrow \quad \varphi = \arcsin(0.571) =$ **34.8°** upstream.

Step 6 — net crossing speed (c). Only the y component remains, i.e. the part of v_{swim} directed across:

$$v_y = v_{swim} \cdot \cos \varphi = 70 \cdot \cos 34.8° = 70 \cdot 0.821 =$$ **57.4 µm/s**

Check. By Pythagoras $v_y = \sqrt{(70^2 - 40^2)} = \sqrt{3300} = 57.4$ µm/s: the same value.

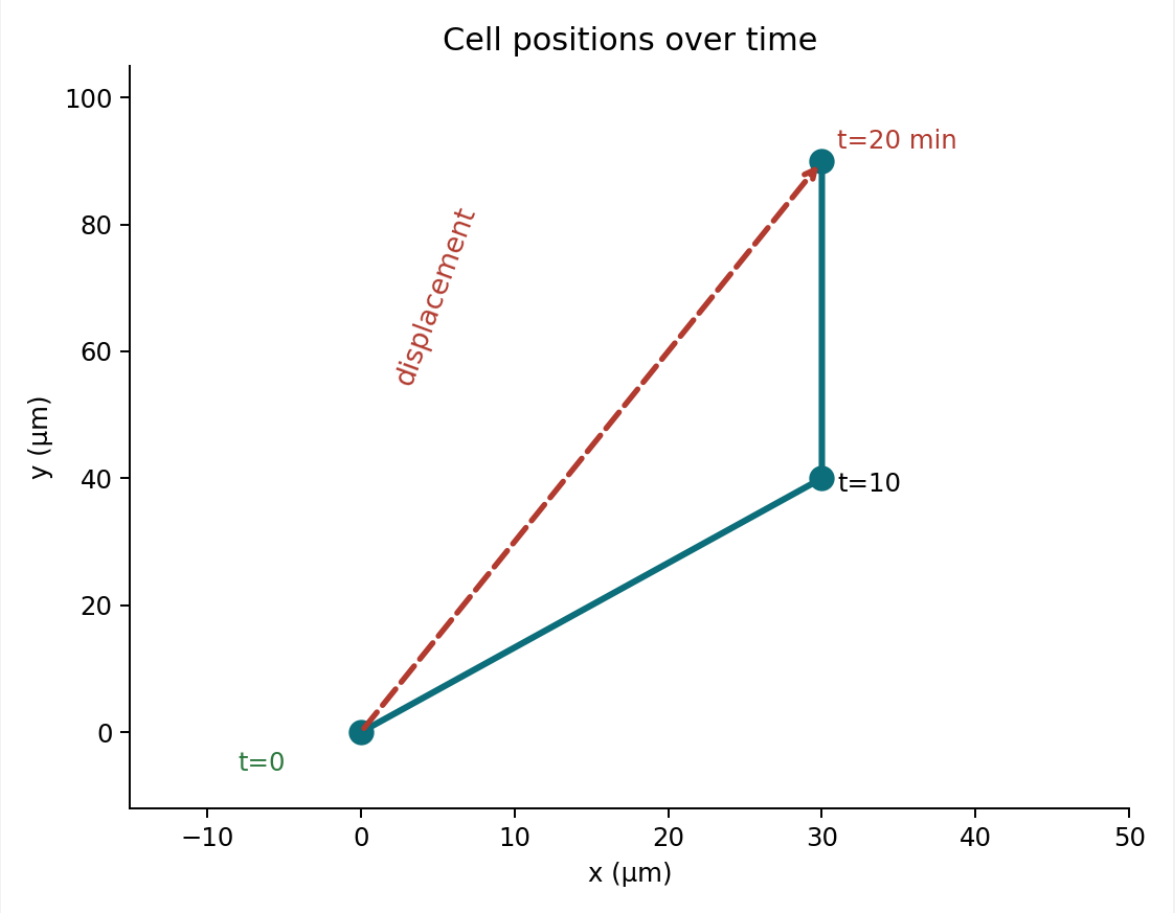
Meaning: 57.4 < 70 µm/s. To compensate for the flow the cell spends part of its speed, so it crosses more slowly (but in a straight line).

Exercise 2 — a cell-migration track

A cell is at (0,0) µm, then (30,40) at t=10 min, then (30,90) at t=20 min.

Find (a) the total path length; (b) the net displacement;

(c) average speed and average velocity; (d) the directedness.



Exercise 2 — solution (a–b)

Step 1 — displacements of the two legs. Compute the displacement vectors $\Delta r = r_{final} - r_{initial}$:

leg 1: $\Delta r_1 = (30 - 0, 40 - 0) = (30, 40)$ µm leg 2: $\Delta r_2 = (30 - 30, 90 - 40) = (0, 50)$ µm

Step 2 — length of each leg (a). The magnitude of a vector with components (x, y) is $\sqrt{(x^2 + y^2)}$:

$|\Delta r_1| = \sqrt{(30^2 + 40^2)} = \sqrt{(900 + 1600)} = 50$ µm $|\Delta r_2| = \sqrt{(0^2 + 50^2)} = 50$ µm

Step 3 — total path length (a). The path length is the sum of the lengths of the legs (a scalar, always positive):

path = 50 + 50 = **100 µm**

Step 4 — net displacement (b). Only the starting and ending points matter, not the route: $\Delta r = (30, 90) - (0, 0) = (30, 90)$ µm. Its magnitude:

$|\Delta r| = \sqrt{(30^2 + 90^2)} = \sqrt{(900 + 8100)} = \sqrt{9000} =$ **94.9 µm**

Note: $|\Delta r| = 94.9$ µm is smaller than the path (100 µm), because the trajectory has a turn.

Exercise 2 — solution (c–d)

Step 5 — average speed (c). The average speed uses the path (a scalar); the time interval is Δt = 20 min:

average speed = path / Δt = 100 / 20 = **5.0 µm/min**

Step 6 — average velocity (c). The average velocity is a vector: net displacement divided by time. Its magnitude:

$|v_{avg}| = |\Delta r| / \Delta t = 94.9 / 20 =$ **4.7 µm/min**

Direction: tan θ = 90 / 30 = 3 → θ = 71.6° from the x axis.

Step 7 — directionality (d). It is the ratio of net displacement to path length: D = |Δr| / path.

D = 94.9 / 100 = **0.95**

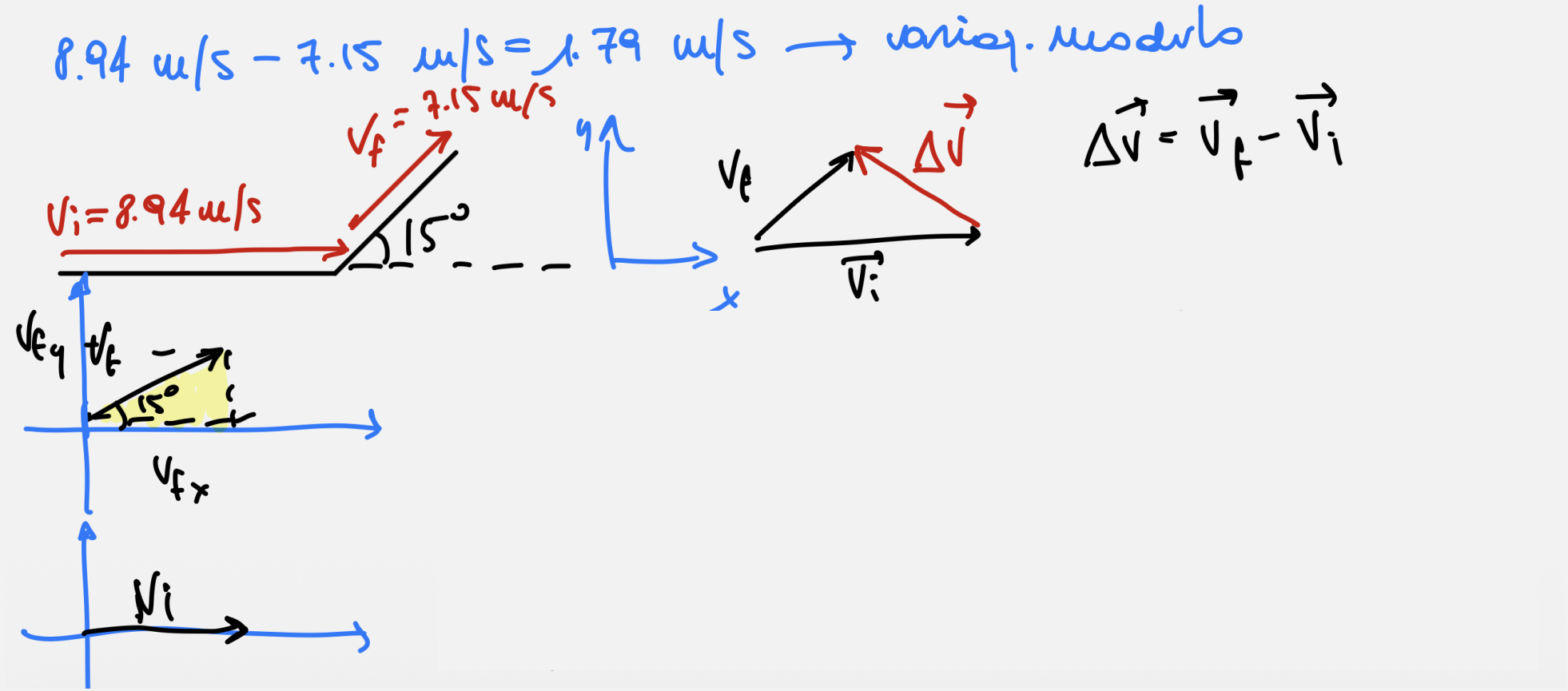
Meaning: D lies between 0 and 1. D = 1 is perfectly straight motion; D near 0 is random motion that returns on itself. Here D = 0.95: the cell moves almost directly.

Exercise 5

A skater is moving along a level road with a speed of 8.94 m/s; after 120.0 s the road begins to incline (at an angle of 15°) and the skater maintains a speed of 7.15 m/s.

(a) What is the change in the skater's velocity?

(b) What is the skater's average acceleration over the 120 s time interval?



Exercise 5 — solution (a)

Step 1 — set up the problem. Velocity is a vector: its change is the vector difference $\Delta v = v_f - v_i$, not the difference of the magnitudes. Take x along the level road and y upwards. Data: $v_i = 8.94$ m/s (horizontal); $v_f = 7.15$ m/s at 15° above the horizontal; $\Delta t = 120$ s.

Step 2 — initial components. The initial velocity is entirely along x:

$v_{ix} = 8.94$ m/s $\qquad v_{iy} = 0$

Step 3 — final components. Project v_f on the axes with $\theta = 15°$:

$v_{fx} = v_f \cos\theta = 7.15 \cdot 0.966 = 6.91$ m/s $\qquad v_{fy} = v_f \sin\theta = 7.15 \cdot 0.259 = 1.85$ m/s

Step 4 — change in velocity (a). Subtract the components:

$\Delta v_x = 6.91 - 8.94 = -2.03$ m/s $\qquad \Delta v_y = 1.85 - 0 = +1.85$ m/s

$|\Delta v| = \sqrt{(2.03^2 + 1.85^2)} =$ **2.75 m/s** $\qquad \theta = \arctan(1.85 / 2.03) = 42°$ above the $-x$ axis (i.e. 138° from $+x$)

Exercise 5 — solution (b)

From part (a): $\Delta v = (-2.03\ ;\ +1.85)$ m/s, $|\Delta v| = 2.75$ m/s, $\Delta t = 120$ s.

Step 5 — average acceleration (b). $a_{avg} = \Delta v / \Delta t$, component by component:

$a_x = -2.03 / 120 = -0.0169$ m/s² $a_y = 1.85 / 120 = +0.0154$ m/s² $|a| = 2.75 / 120 =$ **0.023 m/s²**

Step 6 — check. The difference of the magnitudes is 8.94 − 7.15 = 1.79 m/s (change in speed): it differs from $|\Delta v| = 2.75$ m/s because the direction changes too. The average acceleration has the same direction as Δv (backwards and upwards).

Meaning: the acceleration is small (≈ 0.002 g) because the change takes 120 s; but the velocity changes both in magnitude and in direction, and must be treated as a vector.