Introduction to Multi-Attribute Utility Theory (MAUT)

Many decisions for both individuals and organizations often take into account multiple objectives. From the purchase of a telephone to the choice of the most appropriate localization of an office, decision makers’ choices depend on many different objectives. Often, the multiobjective nature of decisions is revealed by assertions like “we are willing to pay a little bit more to gain the easiness of use and the higher memory of telephone A instead of that of telephone B” in the case of a telephone purchase; or “we agree to increase a little the distance from the city center if, in return, this gives the possibilities of getting closer to our best customers, or if this can reduce the time spent in traffic by our employee”. These statements involve tradeoffs between different objectives of the decision maker.

The first attempts of multiple objective decision aiding date back to Raiffa and Edwards, which gave birth to Decision Analysis. In these works, the decision maker’s preferences are represented numerically on the set of all possible choices using a numerical function called a utility function (or “utility” for short). The key idea of this approach lies in the fact that, after a utility function has been constructed in a simple decision context, it can be used to assign scores or utilities to all potential actions (i.e., the possible choices) that the decision maker faces. Then, these scores can be used to rank the actions from the least desirable to the most desirable one. However, the very fact that such scores can be constructed requires different kinds of conditions to hold, from coherence.

Modeling preferences is generally a trivial task. Given the set $X$ of all possible choices, it amounts to represent the decision maker’s preferences by a binary relation $\succeq$ defined on $X \times X$. But, in practice, manipulating directly the relation $\succeq$ for decision aiding tasks is often neither easy nor efficient. For instance, storing in extension the set $S$ of all pairs $(x, y)$ such that $x \succeq y$ may be impossible in complex situations due to the huge number of such pairs, and also determining the most preferred elements can be very time consuming. This explains why, in practice, instead of using directly $\succeq$ for decision aiding, preferences are often represented numerically through so-called utility functions — or utilities for short — and the latter are used for decision aiding. The idea underlying utility functions is quite simple: these are functions $u: X \rightarrow \mathbb{R}$ attaching to each object of $X$ a real number such that the higher the preferred the object. More formally, this amounts to:

$$
\text{for all } x, y \in X, \quad x \succeq y \iff u(x) \geq u(y).
$$

In practical situations, decision makers have multiple — often contradictory — objectives in mind when making their decisions. This leads to describe the possible consequences using various attributes, that is, the set of consequences is a multidimensional space. For example, a decision maker wishing to buy a new car may have as a choice set $X = \{\text{Opel Corsa, Renault Clio, Fiat Punto}\}$, but if the choice criteria (the attributes) are, among others, the engine size, the brand, and the price of the car, then set $X$ can also be described as $X = \{(1.2L; \text{Opel}; \€11,400), (1.2L; \text{Renault}; \€11,150), (1.1L; \text{Fiat}; \€11,600)\}$. Any utility function over this set thus satisfies the following equation:

$$
\text{for all } x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X, \quad x \preceq y \iff u(x_1, x_2, x_3) \leq u(y_1, y_2, y_3)
$$

This is precisely the multi-attribute utility function. Of course, the meaning of the attributes of relation $\succeq$ heavily depends on the domain of application. For instance:

- in consumer theory, the attributes represent the amounts of some commodity and, for any $x, y \in X$, $x \preceq y$ means that, from the decision maker’s point of view, commodity bundle $y$ is at least as good as $x$;
- in producer theory, $x \in X$ is a vector of inputs and $x \preceq y$ means that $y$ provides at least
as much output as x. The utility function is then called a “production function”;

• in welfare theory, x is an allocation or a social situation. Each attribute represents the wealth of an agent or a player, and x ≤ y means that the wealth of group y is greater than or equal to that of group x;

• in medical decision making, the attributes may represent the level of quality of life that can be expected after undergoing some medical treatment, the expected number of years living at this level of quality of life, and eventually other living parameters.

In general, in each new situation there exists an appropriate set of attributes that can be exhibited in practice (the so-called “structuring of objectives”).

**Example: Teramo and a new electricity generating station.** The City of Teramo is considering four different sites (A, B, C and D) for a new electric power generating station. The objectives of the city are to:

1) minimise the cost of building the station;

2) minimise the acres of land damaged by building it.

Factors influencing the objectives include the land type at the different sites, the architect and construction company hired, the cost of material and machines used, the weather, etc.

Costs, however, are estimated to fall between €15 million and €60 million; between 200 and 600 acres of land will be damaged.

The possible consequences of the decision alternatives are captured by two attributes. We therefore need to determine a two-attribute utility function: \( u(\text{Cost}, \text{Acres}) \) (or \( u(C, A) \)).

Let \( X_1, \ldots, X_n \), with \( n \geq 2 \), be the sets of attributes (\( X_i \) denotes the set of possible values for the \( i \)th attribute) associated with a consequence of a decision problem. Assume, also, that \( X \subseteq X_1 \times X_2 \times \cdots \times X_n \). The utility of the consequences \( (x_1, \ldots, x_n) \in X \) can be determined through:

• a **direct assessment**, that is by estimating directly the combined utility \( u(x_1, \ldots, x_n) \) over all the given values of the \( n \) attributes;

• a **decomposed assessment**, that is by first estimating the \( n \) attribute-specific utility functions \( u(x_i) \) for the actual values of the attributes, and then compute the overall utility function by combining such single utilities

\[
u(x_1, \ldots, x_n) = f(u_1(x_1), \ldots, u_n(x_n)).
\]

For a direct assessment of the utility overall function, in terms of an interval scale, one of the best strategies is to assign the extremals of such an interval as the two utilities to the worst and the best consequence; then, given this, estimate the utilities of all the intermediate consequences.

For example, suppose we want to assess the utility function of a decision problem that we can model with two attributes, that is \( X \subseteq X_1 \times X_2 \). Suppose, also, that the worst consequence in the decision problem is the couple \( (x_{1\text{worst}}, x_{2\text{worst}}) \), while the best is \( (x_{1\text{best}}, x_{2\text{best}}) \). Then, we can estimate the utility of any consequence \( (x_1, x_2) \in X \) by implementing the following steps:

1) assign 0 to the utility of the worst consequence, and 1 to the best one. That is, assign \( u(x_{1\text{worst}}, x_{2\text{worst}}) = 0 \) and \( u(x_{1\text{best}}, x_{2\text{best}}) = 1 \);

2) estimate for each consequence \( (x_1, x_2) \) what is the probability \( p \) that makes the decision maker indifferent between the two lotteries

\[
\begin{array}{c}
\text{1.0} \\
\left(x_1, x_2\right)
\end{array} \quad \text{and} \quad \begin{array}{c}
p \\
\left(x_{1\text{best}}, x_{2\text{best}}\right)
\end{array}
\]

\[
\begin{array}{c}
1-p \\
\left(x_{1\text{worst}}, x_{2\text{worst}}\right)
\end{array}
\]
3) assign the value of $p$ to the utility of the consequence $(x_1, x_2)$ the lotteries are concerned with, i.e. $u(x_1, x_2) = p$ (this is obtained very simply by applying the EU criterion).

However, in order to find a good representation through direct assessment, utilities should be assessed for a suitable number of points. And this, if the attributes and their values are both numerous, can be extremely difficult to make. As preferences, in this context, have to be considered in relation to the all the possible combinations of values for all the attributes, crucial notions are the *conditional* preference and indifference. Roughly, the conditional preference expresses the idea that when all but one attributes are taken fixed to one of their possible values, it is possible to reveal one’s preferences with regard to the attribute left free. More formally, in the case of just two attributes X and Y,

- a consequence $x_1$ is *conditionally preferred* to $x_2$ given $y_0$ if and only if $(x_1, y_0)$ is preferred to $(x_2, y_0)$ (i.e., $(x_1, y_0) \preceq (x_2, y_0)$).

The conditional indifference is analogously defined:

- a consequence $x_1$ is *conditionally indifferent* with $x_2$ given $y_0$ if and only if $(x_1, y_0)$ is indifferent to $(x_2, y_0)$ (i.e., $(x_1, y_0) \sim (x_2, y_0)$).

### Decompositions of utility functions

In practice, the effective construction of overall utility functions raises numerous problems. In fact, although the construction of single-attribute utility functions is generally quite easy, that of multi-attribute utility functions is usually very hard to perform due to the complexity of the problem which increases dramatically with $n$, and to the cognitive limitations of decision makers. Hence, the usual requirement is that they be decomposable as a simple combinations of single-attribute utility functions, which are more easily constructed. These decompositions reflect specific independence properties that attributes may satisfy. Consider, for instance, the case of someone wishing to buy a desktop computer. The attributes of interest are the brand of the computer, its processor, the storage capacity of its hard drive, the size of its LCD display, its memory amount and, of course, its price. One can easily understand why the decision maker should not have too much trouble comparing tuples (Dell; core duo 2GHz; 120GB; 17”; 2GB; €700) and (Apple; core duo 2GHz; 120GB; 17”; 2GB; €700) as these computers differ only by their brand. On the contrary, from a cognitive point of view, it is much more difficult to compare (Dell; 3GHz; 120GB; 24”; 1GB; €800) with (Apple; core duo 1.8GHz; 200GB; 19”; 2GB; €600) as these computers have very different features. This explains why it is usually not possible to construct directly a utility function representing the decision maker’s preferences. Rather, it is more efficient to construct a special form of this function the construction of which will be cognitively more “feasible”.

Several such forms have been studied in the literature, the main ones being described below with their context of application.

1. The *additive* decomposition: there exist some functions $u_i: X_i \rightarrow R$ such that $u(x_1, \ldots, x_n) = \sum_{i=1}^{n} k_i u_i(x_i)$.

   This decomposition can be used both in decision making under certainty and for the EU context. In the case of only two attributes, the additive decomposition takes the form $u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2)$ where the $k_i$'s are the weights of the single-specific utilities in the overall function, and $k_1 + k_2 = 1$. 
2. The multiplicative decomposition: there exist some functions $u_i: X_i \to R$ such that

$$u(x_1, \ldots, x_n) = \prod_{i=1}^{n} u_i(x_i).$$

The multiplicative and the additive decompositions are closely related (the additive is obtainable through a logarithmic transformation of the multiplicative, by assuming the $u_i$'s are such that $u_i \geq 0$). In the case of two attributes, and under an appropriate transformation, this decomposition takes the form $u(x_1, x_2) = u_1(x_1)u_2(x_2)$.

3. The multilinear decomposition (also called polynomial or multiplicative-additive): there exist functions $u_i: X_i \to R$ and, for every $j \in J$, where $J$ is the set of subsets of \{1, \ldots, n\}, there exist some $k_j \in R$ such that

$$u(x_1, \ldots, x_n) = \sum_{j \in J} k_j \prod_{k \in j} u_k(x_k).$$

This decomposition has been studied for decision making under certainty and for EU situations. In the case of just two attributes, the additive decomposition takes the form $u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2) + k_{12} u_1(x_1)u_2(x_2)$ where the $k_j$'s are the weights of the single-specific utilities in the overall function, and it holds that $k_1 + k_2 + k_{12} = 1$.

Note, however, that such decomposition can be applied only when certain independence conditions between the attributes are met. When these conditions are satisfied, the multi-attribute utility function can be expressed in one of these simplified ways. In the following, some typical conditions are explained, together with the utility function decomposition they entail.

Additive independence

Use of an additive utility function is justified given the assumption of additive independence [AI]. Two attributes $X$ and $Y$ are additive independent if preferences for lotteries over $X \times Y$ can be established by comparing the values one attribute at a time. More formally,

two attributes $X$ and $Y$ are additive independent when the paired preference comparison of any two lotteries, defined by two joint probability distributions on $X \times Y$, depends only on their marginal distributions.

Practically, this means that two attributes $X$ and $Y$ are additive independent if and only if for an arbitrary pair $(x_A, y_A)$, and for all pairs $(x, y)$, the decision maker is indifferent between the two lotteries

- **Lottery A:** $0.5 \cdot (x_A, y)$
- **Lottery B:** $0.5 \cdot (x_A, y_A)$

For example, given the decision problem of buying a car and considering the attributes color (red/black) and style (sports car/SUV), if the decision maker is indifferent between a 0.5 probability of a red sports car and 0.5 probability of a black SUV, and the 0.5 probability of a red SUV and 0.5 probability of a black sports car, then color and style are additive independent.

Note that, in each of the two lotteries A and B there a 50% probability of getting either $x_A$ or $x_i$ and a 50% probability of getting either $y_A$ or $y_i$. The only difference is how the different values of $X$ and $Y$ are combined. From this, it is clear that additive independence is a symmetric property.

Moreover, this is the strongest independence property that can be obtained for a multi-attribute utility function. In fact, for example in the case of two attributes that are additive independent, the overall utility function can be expressed as
\[ u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2) \]

where
- \( u(x_1, x_2) \) is normalized by \( u(x_{1\text{worst}}, x_{2\text{worst}}) = 0 \) and \( u(x_{1\text{best}}, x_{2\text{best}}) = 1 \);
- each \( u_i(x_i) \) is the specific utility function of \( X_i \) normalized by \( u_i(x_{i\text{worst}}) = 0 \) and \( u_i(x_{i\text{best}}) = 1 \);
- the weights \( k_i \) are positive constants such that:
  \[ k_1 + k_2 = 1, \quad k_1 = u(x_{1\text{best}}, x_{2\text{worst}}) \quad \text{and} \quad k_2 = u(x_{1\text{worst}}, x_{2\text{best}}) \]

Analogously, if a decision problem has three additive independent attributes, its overall utility function has the following form:
\[ u(x_1, x_2, x_3) = k_1 u_1(x_1) + k_2 u_2(x_2) + k_3 u_3(x_3) \]

Thus, with additive independence, the overall utility function can be totally separated into additive contributions: one for each attribute-specific utility function.

**Utility independence**

Use of a multilinear or multiplicative utility function is justified under (mutual) utility independence [(M)UI].

Attribute \( X \) is utility independent of attribute \( Y \) if conditional preferences for lotteries over \( X \) given a fixed value for \( Y \) do not depend on the particular value for \( Y \). More formally, if \([X, y]\) represents a conditional lottery over \( X \times Y \) involving consequences over different values of \( X \) combined with a fixed value \( y \) for \( Y \), then

an attribute \( X \) is utility independent of an attribute \( Y \) if and only if

for any lotteries \([X, y_1]\) and \([X, y_2]\) over \( X \times Y \) with \( Y \) fixed to value \( y_b \) we have

\[ [(X, y_1)]_1 \preceq [(X, y_2)]_2 \implies [(X, y_1)]_1 \preceq [(X, y_2)]_2 \quad \forall y_k \text{ of } Y \]

Practically, \( X \) is utility independent of \( Y \) if and only if, for an arbitrary triple \( x_b, x_b, x_G \) with \( x_A \leq x_B \leq x_G \), there exists a probability \( p \) such that the decision maker is indifferent between the two lotteries

\[
\text{Lottery A: } \begin{array}{c}
\circ \quad 1.0 \quad (x_b, y) \\
\end{array} \\
\text{Lottery B: } \begin{array}{c}
\circ \quad p \quad (x_A, y) \\
\circ \quad 1-p \quad (x_G, y) \\
\end{array}
\]

and the indifference remains for all values \( y_i \) of \( Y \). This means that the preferences for lotteries on \( X \) given a value for \( Y \), do not depend on the particular level of \( Y \). Hence, if \( X \) is utility independent of \( Y \), if lotteries A and B are equally preferable (i.e., indifferent), then also the following lotteries C and D are indifferent, for any choice of \( y_k \) in \( Y \).

\[
\text{Lottery C: } \begin{array}{c}
\circ \quad 1.0 \quad (x_b, y) \\
\end{array} \\
\text{Lottery D: } \begin{array}{c}
\circ \quad p \quad (x_A, y) \\
\circ \quad 1-p \quad (x_G, y) \\
\end{array}
\]

For example, in a decision problem regarding the buying a car, consider a two attribute sets of color (white/black/grey) and style (sports car/SUV). Suppose the decision maker is indifferent between the certainty of a grey sports car and a 50% chance of a white sports car with a 50% chance of a black sports car. If the decision maker is also indifferent between the certainty of a grey SUV and a 50% chance of a white SUV with a 50% chance of a black SUV, then the attribute of color is utility independent from the attribute of style.

Attribute \( X_1 \) may be utility independent from attribute \( X_2 \), but not necessarily vice versa. But if \( X_1 \) is utility independent of \( X_2 \) and \( X_2 \) is utility independent of \( X_1 \), then \( X_1 \) and \( X_2 \) are said to be
mutually utility independent. In this case, the overall utility function can be expressed in the form

\[ u(x_1, x_2) = k_1 u_1(x_1) + k_2 u_2(x_2) + k_3 u_1(x_1)u_2(x_2) \]

where

- \( u(x_1, x_2) \) is normalized by \( u(x_{1\text{worst}}, x_{2\text{worst}}) = 0 \) and \( u(x_{1\text{best}}, x_{2\text{best}}) = 1 \);
- each \( u_i(x_i) \) is the specific utility function on \( X_i \) normalized by \( u_i(x_{i\text{worst}}) = 0 \) and \( u_i(x_{\text{best}}) = 1 \);
- the weights \( k_i \) are positive constants such that:
  \[ k_1 = u(x_{\text{best}}, x_{2\text{worst}}), \ k_2 = u(x_{1\text{worst}}, x_{2\text{best}}) \] and \( k_3 = 1 - k_1 - k_2 \)

Analogously, if a decision problem has three mutual utility independent attributes, its overall utility function has the following form:

\[ u(x_1, x_2, x_3) = k_1 u_1(x_1) + k_2 u_2(x_2) + k_3 u_3(x_3) + k_4 u_1(x_1)u_2(x_2) + k_5 u_1(x_1)u_3(x_3) \\
+ k_6 u_2(x_2)u_3(x_3) + (1 - k_1 - k_2 - k_3 - k_4 - k_5)u_1(x_1)u_2(x_2)u_3(x_3) \]

In general, utility independence means that each attribute can be considered with its own specific utility function and the overall utility function of the decision problem, which is a multi-variable function, can be decomposed in a way that uses only those attribute-specific utility functions.

This kind of independence is heavily considered in the literature, and a useful assumption in many cases, as Keeney and Raiffa suggest: "The utility independence assumptions are appropriate in many realistic problems, and they are operationally verifiable in practice". Furthermore, note that the property of utility independence is weaker than additive independence. That is, if two attributes are additive independent, then they must be mutual utility independent.

**Preferential independence**

Attribute \( X \) is preferentially independent of attribute \( Y \) [PI] if the conditional preferences for the values of \( X \), given a fixed value for \( Y \), do not depend on the particular value for \( Y \). More formally,

an attribute \( X \) is preferentially independent of an attribute \( Y \) if and only if for some \( y_i \) and any consequences \( (x_A, y_i) \) and \( (x_B, y_i) \) over \( X \times Y \), we have

\[ (x_A, y_i) \leq (x_B, y_i) \implies (x_A, y_i) \leq (x_B, y_i) \ \forall y_k \text{ of } Y \]

Thus, \( X \) is preferentially independent of \( Y \) when the preferences in the attribute \( X \) do not depend on the preferences in the attribute \( Y \). In other words, changes in the rank ordering of preferences of attribute \( Y \) does not change the preference order of the attribute \( X \).

If \( X \) is preferentially independent of \( Y \) and \( Y \) is preferentially independent of \( X \) then \( X \) and \( Y \) are mutually preferential independent [(M)PI].

In terms of the utility function, when two attributes \( X_1 \) and \( X_2 \) are preferentially independent, their preference structures can be considered separately. This means that the overall utility function can be expressed in the form

\[ u(x_1, x_2) = f(u_1(x_1), u_2(x_2)) \]

where each \( u_i(x_i) \) is the attribute-specific utility function on \( X_i \). Operationally, the decision maker can structure her preferences for the first attribute without worrying about the second, and for the second without worrying about the first.

For example, let’s say the two attributes for a car are color (red/black) and style (sports...
car/SUV). If the decision maker prefers a red SUV over a black SUV, and prefers a red sports car over a black sports car, then the color is preferentially independent of style: red is preferred over black, regardless of style. However, style is not necessarily independent from color. For example, it is possible that the decision maker prefers a red sports car over a red SUV, but prefers a black SUV over a black sports car. In this case, the preference on style is not independent of color.

Preferential independence is weaker than utility independence. That is, if two attributes are utility independent, then they must be preferential independent as well.